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Price Indexes across Space and Time and the Stochastic Properties of Prices

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Abstract The availability of scanner data from large-scale retailers makes the construction of a continuously updated system of price indexes over space and time for an important share of household consumption expenditures possible. However, building a coherent (transitive) system of price indexes across space and time involves issues that are irrelevant for bilateral price indexes or multilateral price indexes only over space. Some of these issues were discussed by Hill (2004), but in my opinion the most important has been ignored. Indeed, it is very likely that the same commodity is differently priced across space, but in the long run the movements of its prices will be similar (stable) in space. So it is quite natural to ask price indexes for pairs of space situations not to diverge over time if the prices of each single commodity in the basket remain approximatively pairwise proportional in the two sites. In this work, we give a definition of the test of *stability preservation*, starting from the stochastic properties that panels of price time series seem to obey to. Then, many different approaches to the construction of the system of indexes are analysed in order to identify those that pass the test. The selected systems are applied both to simulated and to real-world data collected in four supermarkets located in the city of Milan for a time span of 24 months.

Key words: Price index, multilateral comparison, purchasing power parity, scanner data, cointegration, stability.

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1 Introduction

Many institutions responsible for the production of official price statistics are moving towards automatic procedures for surveying the relevant data. The first Italian city to start this process was the City of Milan, which since the end of the Nineties receives price data directly from the databases of large-scale retailers.

This work is part of a research project in which the Statistical Office of the City of Milan aims at expanding the information base gathered from large-scale retailers in order to have actual (cash) sale prices and quantities. At the moment, few supermarkets are providing us with monthly sales data which include total monthly sold values and quantities for every single product on their shelves.

The availability of this kind of data is extremely valuable since it allows the comparison of the cost of living across space and time. Supermarket products are, indeed, rather homogeneous over both dimensions (space and time) and represent an important share of personal consumptions (in Italy 77% of food and 88% of beverage and grocery expenditures according to AC Nielsen).

A natural problem that arises when a system of space-time price indexes is to be computed is the choice of the formulas. Indeed, while there is a general consensus on the choice of bilateral indexes, the debate on multilateral comparisons is still open and will probably remain such in the future. In fact, it is well known that in a system of multilateral price-quantity indexes the property of transitivity, essential for building a coherent system, is incompatible with the property of characteristicity and consequently with the proportionality of either the price indexes or their respective cofactor. This amounts to the impossibility of factorizing all the value indexes in a coherent system of proportional price indexes times proportional quantity indexes. This is maybe not surprising as index numbers are an extremely synthetic way of summarizing huge masses of information. For thorough discussions of this theme the reader should refer to Balk (1996); van Veelen (2002); Balk (2008, Ch.7).

The consequence of the above consideration is that whoever wants to build a transitive system of price and quantity indexes needs to express a preference on the properties the system and the single pairs of price and quantities indexes should satisfy. Now, for our type of data (supermarket scanner data) it is natural to ask price indexes rather than quantity indexes to pass all possible tests. Indeed, price indexes would contain important information for comparing the cost of living in different places, while quantity indexes would just indicate that a supermarket sells more or less than another supermarket. On the contrary, if the interest is comparing the wealth of a country with respect to other countries through their national accounts, then quantity indexes are probably more informative. The fact that products may change through time and space brings about further considerations that have to be accounted for in the construction of the system of indexes.

In this paper, I address the problem of building a coherent system of space-time price indexes that should be the best possible compromise, when all the issues that our data pose have been considered. The novelty of the approach I pursue in this work is the consideration also of the stochastic properties that prices and sold quantities of each product seem to conform to in real-world data. In this light, the new properties of *cointegration-preservation* and the slightly more general *stability-preservation* are defined and discussed. I do not seek with this paper the production of a complete theory of cointegration- or stability-preserving indexes, since, as discussed in section 3, “less linear” definitions of integrated and cointegrated processes are needed (Ermini and Granger, 1993; Granger, 1995) given the non-linearity of index number functions. However, in Section 4 I try to give some first theoretical results basing the definition of stability-preservation on (strict) stationarity and (strong) mixing of prices and indexes.

2 Existing literature

While there is some literature on space-time harmonization of official CPI and PPP statistics (Rao, 2001; Ferrari et al., 2005; Eurostat, 1995, Sec.7), the issue of building a coherent system of indexes over space and time starting from raw data (to the best of my knowledge) has been considered only by Hill (2004). Martini and Zavanella (1995) considered the problem as well, but supposing that quantities are not available and found that the only solution compliant with the axiomatic theory is a system of direct Jevons indexes (i.e. geometric mean of price ratios).

Thus, the starting point of this work is the article by Hill (2004), but as will stand clear from the next sections, my recommendations concerning the appropiate formulas for the index system will differ significantly from Hill’s.

Hill (2004) considers the following classes of multilateral formulas

- Geary-Khamis (Geary, 1958; Khamis, 1972),
- GEKS (Gini, 1924; Eltetö and Köves, 1964; Szulc, 1964),
- Minimum Spanning-Tree methods (Martini, 1992; Zavanella, 1996),
- the Weighted Country Product Dummy (WCPD) method (Rao, 2005),

under different graph configurations and possible alternatives to join sub-graphs into a single graph. The different formulas and graph configurations are evaluated according to the following criteria:

Temporal Fixity (TF): old indexes are not affected when information concerning a new time period is added;
Spatial Fixity (SF): old indexes are not affected when information concerning a new country is added;
Temporal Consistency (TC): the temporal indexes for each country do not depend on the other countries in the comparison;
Spatial Consistency (SC): spatial results for each time point do not depend on the other time periods in comparison;
Temporal Displacement (TD): maximum temporal displacement of all the bilateral spatial comparisons, where the temporal displacement of a bilateral comparison is the time span of the data used to compute it; if there are T time points, the temporal displacement ranges between 0 (only contemporaneous data used) and $T-1$ (data from time T and form time 1 used).

As Hill notices, in a transitive system of indexes temporal and spatial consistency cannot coexist if quantities are to be taken into account.

In his considerations on the choice of a method for space-time comparisons Hill suggests "that one should prefer methods that maintain temporal fixity and temporal consistency". In order to reduce temporal displacement Hill suggests that temporal consistency could be periodically broken in order to reconcile spatial and temporal price indexes and proposes different method to do this. It is to be stressed that Hill draws his conclusions with in mind the comparison of European Union countries based on the Harmonized Indexes of Consumer Prices and OECD spatial price indexes. So his ideal setup is somewhat difference from mine.

As the next sections will show, according to the nature of our data the property of time consistency turns out to lead to indexes with very undesirable properties. Furthermore, in addition to the families of multilateral indexes considered by Hill, also the performances of the method used by the Economic Commission for Latin America and the Caribbean (ECLAC, 1978) and Gerardi's (1982) Unit-Country-Weight (UCW) system will be analyzed.

3 Price behaviour and price indexes

Before being able to express a rational preference for a particular transitive system of multilateral indexes, it is important to consider how prices and quantities tend to behave both in space and time. The best way to describe the behaviour of prices and quantities in time is probably through stochastic models.

A reasonable model for the dynamics of (log) prices in discrete time is the following integrated process

$$\log p_{n,k,t} = \log p_{n,k,t-1} + \delta_{n,k} + \eta_{n,k,t}, \tag{1}$$

where $p_{n,k,t}$ denotes the price of commodity $n \in \{1,\dots,N\}$ in country $k \in \{1,\dots,K\}$ at time $t \in \{1,\dots,T\}$, $\delta_{n,k}$ is a time-invariant drift parameter and $\{\eta_{n,k,t}\}_{t=1,\dots,T}$ is a mean-zero stationary sequence. Notice that, since $\log(1+x) \approx x$ for small x, then, for moderate price increases,

$$\frac{p_{n,k,t} - p_{n,k,t-1}}{p_{n,k,t-1}} \approx \log\frac{p_{n,k,t}}{p_{n,k,t-1}} = \delta_{n,k} + \eta_{n,k,t},$$

and $\delta_{n,k}$ turns out to be approximately the mean increment rate of the price of product n in country k with respect to one time unit. If the hypothesis of a constant mean increment rate is too strong, the drift $\delta_{n,k}$ may be made time-dependent with the possible specification

$$\delta_{n,k,t} = \delta_{n,k,t-1} + \zeta_{n,k,t}, \tag{2}$$

where $\zeta_{n,k,t}$ is, again, a mean-zero stationary sequence. The process (1) is commonly referred to as integrated process of order 1, or in short I(1), while the pair (1)-(2)

forms an integrated process of order 2, or I(2). More generally, a process I(d) is a nonstationary process that needs (at least) d differences to become stationary.

Just as an illustrative example, let's consider the monthly prices (in US$ per MT[2]) of the Thai rice (B garde, FOB), which is easy to have access to, being frequently traded in international commodity markets (Figure 1). Applying unit root tests[3] to the log-prices we cannot reject the null of a unit root at any commonly used size, while the same tests applied to the increments (differences) of log-prices reject the hypothesis of integration. Indeed an ARIMA(1,1,0) process seems to fit the rise log-prices quite well.

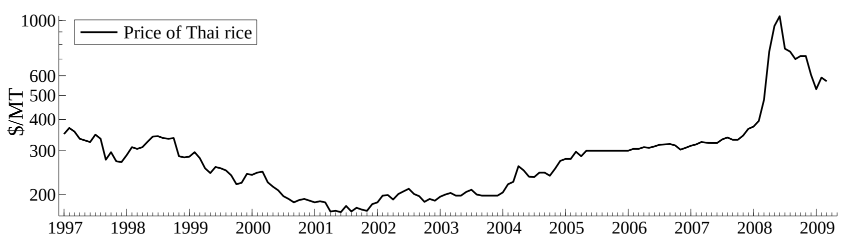

Fig. 1 Price of Thai rice in log scale from Jan. 1997 to Mar. 2009.

If we consider the nature of supermarket data within a country, then it is very likely that the same product across different stores will have in the long-run approximately proportional prices[4]. In fact it is hard to believe that the prices of the same good across different stores may drift apart without bound, while it not strange to observe the same goods differently priced across a country or a city. Turning these concepts in formulas, we have

$$p_{n,k,t} \approx b_{n,k,j} p_{n,j,t} \quad \Leftrightarrow \quad \log p_{n,k,t} \approx \beta_{n,k,j} + \log p_{n,j,t} \tag{3}$$

with $k \neq j$, $\beta_{n,k,j} = \log b_{n,k,j}$. This condition may be stated in a more statistically meaningful fashion as

$$\log p_{n,k,t} = \beta_{n,k,j} + \log p_{n,j,t} + \varepsilon_{n,k,t}, \tag{4}$$

[2] MT stands for *measurement ton* a unit of volume used for measuring the cargo of a ship, truck, train, or other freight carrier, equal to exactly 40 cubic feet, or approximately 1.1326 cubic meters.

[3] The ADF, and Elliot et al.'s (1996) DF-GLS point-optimal tests were applied. The stationarity test KPSS leads to the same conclusions.

[4] The *Law of One Price* implies that prices of tradable goods should be equal in the long run, but in order to allow some kind of *intra-country Balassa-Samuelson effect* we relax the identity to proportionality.

where $\varepsilon_{n,k,t}$ is a mean-zero stationary process. If the price time series are well described by the processes (1)-(2) above, then this condition (4) amounts to say that the prices of good n in the two countries are *cointegrated*[5]. It is trivial to prove that cointegration between pairs of price series is a transitive property: if A is cointegrated with B and B is cointegrated with C, then A is cointegrated with C.

Now, if all the commodities in two countries of a panel are cointegrated, then the corresponding space index should be stationary with respect to the time parameter t. In fact, if this does not happen, the index reveals price divergence even though all price-pairs are not diverging.

The aforementioned property for space-time index numbers could be named *cointegration-preservation* if we are ready to accept the integration/cointegration approach, but could also be named *stability-preservation* if we want to leave the concepts of divergence and co-divergence of prices open, where by divergence it is meant that the price of a commodity does not have an *attractor*[6] and by co-divergence we mean that the ratio of the prices of the same good in two countries does have an attractor.

Even if we concentrate on the somewhat smaller world of cointegration-preserving indexes, the development of a complete theory is rather complex, as the concepts of integration and cointegration are intrinsically linear, while price index formulas are not necessarily linear in the logarithm of price ratios. The analysis of nonlinear functions of integrated and cointegrated processes needs the development of alternative concepts and tools and goes beyond the scope of this paper. Ermini and Granger (1993) and Granger (1995) are among the very few papers I am acquainted with that deal with this issue, and could be a starting point for building a complete theory of cointegration-preserving indexes. However, in the next Section we try to attack the problem from a theoretical point of view, giving a formal definition of stability-preservation and few results that should shed some light on the conjectures we are making in this section.

From equation (4) it is clear that in order to recover the discrepancy between $p_{n,j,t}$ and $p_{n,k,t}$ the index should be built as a function of the elementary price ratios

$$\frac{p_{n,k,t}}{p_{n,j,t}} = b_{n,k,j}\exp(\varepsilon_{n,k,t}),$$

while other type of functions of the the N prices will in general drift apart even when all the prices of the same commodity are cointegrated across country pairs. Surprisingly this condition seems to be only sufficient for the construction of cointegration-preserving indexes, in fact the ratio of values evaluated with fixed quantities

[5] The reader not acquainted with the concept of cointegration should refer to any recent text on time series econometrics. A thorough and mathematically rigourous treatment of cointegration in the framework of vector autoregressive processes may be found in Johansen (1995)

[6] In order to leave the concept open enough we can take the definition of Granger (1995): “a process may be said to have an attractor if there is some mechanism that produces an inclination to return to some value – usually its mean”.

$$\frac{\sum_{n=1}^{N} p_{n,k,t} q_n}{\sum_{n=1}^{N} p_{n,j,t} q_n} \tag{5}$$

seems to enjoy the property, even though a formal proof of this fact can be sought only with a more specific definition of *attraction* to a fixed value or *mean-reversion* (cf. Section 4).

Figure 2 depicts simulated time series and their ratios for every time point. Three pairs of integrated and (country-wise) cointegrated (log-price) time series are generated for $n = 1,2,3$ and $t = 1,\dots,100$ from

$$\log p_{n,1,t} = \sum_{s=1}^{t} \varepsilon_{n,1,s}, \qquad \varepsilon_{n,1,s} \sim \text{NID}(0,1) \tag{6}$$

$$\log p_{n,2,t} = \log p_{n,1,t} + \eta_{n,t}, \qquad \eta_{n,t} \sim \text{NID}(0,1) \tag{7}$$

where $\text{NID}(\mu,\sigma^2)$ stands for normally independently distributed with mean μ and variance σ^2. Panel a) of Figure 2 shows the three ratios $p_{n,2,t}/p_{n,1,t}$ on a log scale. Since the three ratios tend to move around an attractor, so should the price index between the two countries. Panel b) represents the index defined in equation (5) with quantities vector $\boldsymbol{q} = (1,10,100)'$. Panel c) reports the values of an index built as ratio of values where the quantities for the numerator are $\boldsymbol{q}_2 = (100,1,10)$ and those for the denominator are $\boldsymbol{q}_1 = (1,10,100)$. While the index in panel b) tends to move around the attractor 1, the index in panel c) does not seem to enjoy this property.

The index-number-theory competent reader would certainly object that an index of the type depicted in panel c) is not a proper index (at least axiomatically), but many economists test the *purchasing power parity theory* using this type of indexes. In fact, in many empirical papers unit root tests are applied to (the logarithm of) real exchange rates (RER) built using consumer price indexes (CPI) and nominal exchange rates of country pairs: $\text{RER}_t = E_t \times \text{CPI}_{2,t}/\text{CPI}_{1,t}$, with E_t nominal exchange rate. Now, without loss of generality suppose that the exchange rate is constantly equal to one, then it is straightforward to see that RARs are built exactly as those indexes of panel c)[7]. Applying a unit root test[8] to our simulated data, we get a rejection (mean-reversion) for panel b) and a non-rejection (integration) for panel c) data.

[7] Recall that the CPI index for country k has generally the form

$$\text{CPI}_k = \sum_n \frac{p_{n,k,t}}{p_{n,k,0}} w_n$$

for some weights $\{w_n\}$. Thus, if log-prices are modelled as integrated processes then $\log p_{n,k,t} = \sum_{s=1}^{t} \varepsilon_{n,k,s} + \log p_{n,k,0}$, and $\log(p_{n,k,t}/p_{n,k,0})$ is an integrated process that starts form zero (at time $t = 0$), exactly as in our simulation.

[8] A Dickey-Fuller test was applied imposing non deterministic regressors, since under the alternative of mean reversion the mean of the log of the price indexes should be 0, being the price pairs almost identical.

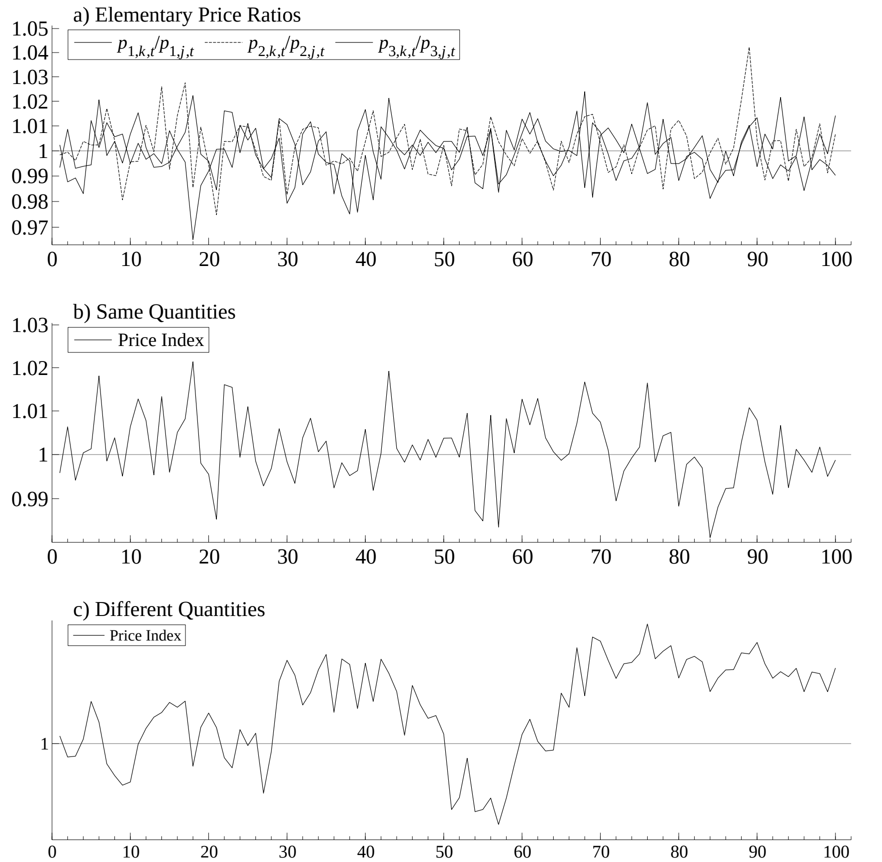

Fig. 2 a) price ratios, b) ratio of values based on same quantities, c) ratio of values based on different quantities. All values are on a log-scale.

Thus, the first (empirical) result of the above reasoning is that the so called *purchasing power parity puzzle*, that is, the fact that real exchange rates turn out to be integrated or at least too persistent if compared to what the theory prescribes, may be ascribed to the (wrong) choice of the index number formula (ratio of CPIs) for building real exchange rates. Even though this result may be considered a byproduct of our research, its relevance could be far-reaching if the reader considers that the exact sequence of words “purchasing power parity puzzle” produces 7650 entries in Google and 1810 in Google Scholar.

Recall that in a system of multilateral price indexes the properties of transitivity and characteristicity are incompatible. Since we want to preserve transitivity, the indexes of the system we are going to build won’t be characteristic. From index number theory (Martini, 2001, Section 6.2), we know that if we have the price vectors of two situations under comparison and *non-characteristic values* $\{v_n\}$, then the only index that satisfies proportionality (PR), commensurability (C), homogeneity (H), monotonicity (M) and base reversal (B) has form

$$\prod_{n=1}^{N} \left(\frac{p_{n,k}}{p_{n,j}}\right)^{w_n}, \qquad \text{with} \quad w_n = \frac{v_n}{\sum_{n=1}^{N} v_n}. \tag{8}$$

If, instead, only non characteristic quantities $\{q_n\}$ are available, then the only index with properties PR,C, H, M, B has form

$$\frac{\sum_{n=1}^{N} p_{n,k} q_n}{\sum_{n=1}^{N} p_{n,j} q_n}. \tag{9}$$

It is interesting to notice that, from our above reasoning, these two type of indexes are those that appear compliant with the property of cointegration preservation. The fact that the values v_n and the quantities q_n may evolve with time further complicates the theoretical analysis of cointegration-preserving or stability-preserving indexes.

4 Some theoretical results

There are many alternative way to attack the problem of stability-preserving indexes that has been outlined in the previous sections. Here, I provide some preliminary results based manly on probabilistic arguments, but alternative results could be based on nonlinear dynamic systems and stability theory as well.

From the above discussion, the main features of a stability-preserving index could be formalized in the following assumption.

Assumption 1 (stationarity and mean reversion of simple price ratios). *For each product* $n \in \{1, \dots, N\}$ *and for the two countries* j *and* k *let*

$$\frac{p_{n,k,t}}{p_{n,j,t}} = b_{n,k,j} \exp\{\eta_{n,kj,t}\},$$

with $b_{n,j}$ *positive real constants and* $\boldsymbol{\eta}_t = \{\eta_{1,kj,t}, \dots, \eta_{N,kj,t}\}$ *strictly stationary strong mixing (SSSM) sequence*[9]*, such that*

$$\mathbb{E}\exp\{\eta_{n,kj,t}\} = m_n < \infty \qquad \forall n.$$

[9] Let $\{X_t\}_{-\infty}^{\infty}$ be a (possibly vectorial) random sequence and let $\mathscr{F}_{-\infty}^{t}$ and $\mathscr{F}_{t+m}^{\infty}$ the σ-fields generated by, respectively, $\{\dots, X_{t-2}, X_{t-1}, X_t\}$ and $\{X_{t+m}, X_{t+m+1}, X_{t+m+2}, \dots\}$ and define

$$\alpha_m := \sup_{A \in \mathscr{F}_{-\infty}^{t},\, B \in \mathscr{F}_{t+m}^{\infty}} |\Pr(A \cap B) - \Pr(A)\Pr(B)|.$$

The sequence $\{X_t\}_{-\infty}\infty$ is said to be *strong mixing* if $\lim_{m\to\infty} \alpha_m = 0$. For a rigorous introduction to the theory of mixing random sequences refer to Davidson (1994, Ch. 13-14)

This assumption asks the simple price ratios to have a time invariant mean and to be mean-reverting to that mean. Indeed, while it is not so easy to find a generally accepted definition of mean-reversion, mixing conditions are well defined and imply the mean-reverting behaviour. From an economic point of view, Assumption 1 is the probabilistic version of the (weak) law of one price[10].

Proposition 1 (Sufficient conditions for stability-preserving indexes). *Suppose that Assumption 1 holds, and name $\mathscr{G}$ the class of all measurable functions $\mathbb{R}^{2N} \mapsto \mathbb{R}$ of the random prices $\{p_{n,j,t}, p_{n,k,t}\}_{n\in\{1,\dots,N\}}$ with finite expectation for every t. Then, all the functions belonging to $\mathscr{G}$ with form*

$$g\left(p_{1,k,t}, p_{1,j,t}, p_{2,k,t}, p_{2,j,t}, \dots, p_{N,k,t}, p_{N,k,t}\right) = g\left(\frac{p_{1,k,t}}{p_{1,j,t}}, \frac{p_{2,k,t}}{p_{2,j,t}}, \dots, \frac{p_{N,k,t}}{p_{N,k,t}}\right),$$

are strictly stationary strong mixing.

Proof. The proof is trivial since measurable time-invariant finite-lag functions of strictly stationary strong mixing sequences are strictly stationary strong mixing sequences (for example Davidson, 1994, Th. 14.1). □

As noticed in the preceding section, and as it will appear in the next section, an index with form

$$\frac{\sum_{n=1}^{N} \alpha_i p_{n,k,t}}{\sum_{n=1}^{N} \alpha_i p_{n,j,t}}, \qquad \text{with} \quad \alpha_n \geq 0,\ \forall n, \quad \sum_{n=1}^{N} \alpha_n = 1 \tag{10}$$

seems to preserve mean-reversion under Assumption 1. In order to gain some more insight on this formula, let us consider the following version of the index with just two products

$$\frac{\alpha_1 \exp(\beta_1 + \mu_{1,t} + \eta_{1,t}) + \alpha_2 \exp(\beta_2 + \mu_{2,t} + \eta_{2,t})}{\alpha_1 \exp(\mu_{1,t}) + \alpha_2 \exp(\mu_{2,t})}, \tag{11}$$

where, for $n = \{1,2\}$, β_n are constants, $\mu_{n,t}$ are non-stationary non-mixing processes and $\eta_{n,t}$ are zero-mean SSSM sequences. If $p_{n,k,t} = \exp(\beta_n + \mu_{n,t} + \eta_{n,t})$ and $p_{n,j,t} = \exp(\mu_{n,t})$, then

$$\frac{p_{n,k,t}}{p_{n,j,t}} = \frac{\exp(\beta_n + \mu_{n,t} + \eta_{n,t})}{\exp(\mu_{n,t})} = \exp(\beta_n)\exp(\eta_{n,t}),$$

respects Assumption 1.

First, notice that equation (11), by multiplying and dividing the numerator by $\alpha_1 \exp(\beta_1 + \mu_{1,t} + \eta_{1,t})$ and the denominator by $\alpha_1 \exp(\mu_{1,t})$, can be rewritten as

$$\exp(\beta_1 + \eta_{1,t})\frac{1 + \alpha \exp(\beta + \mu_t + \eta_t)}{1 + \alpha \exp(\mu_t)}, \tag{12}$$

[10] The law of one price ask price ratios to have mean one, here we limit the request to the existence of a positive mean. So, in the long-run, prices can be proportional and not necessarily identical.

with $\alpha = \alpha_2/\alpha_1$, $\beta = \beta_2 - \beta_1$, $\mu_t = \mu_{2,t} - \mu_{1,t}$ and $\eta_t = \eta_{2,t} - \eta_{1,t}$.

The process (12) is the product of a SSSM process and a process that does not seem to share the SSSM property, but that approaches it as μ_t diverges. If we restrict our attention to the following independent Gaussian processes

$$\mu_{n,t} = \delta_n t + \sum_{s=1}^{t} \varepsilon_{n,s}, \qquad \varepsilon_{n,s} \sim \mathrm{NID}(0, \sigma_n^2), \qquad \eta_{n,t} \sim \mathrm{NID}(0, \tau_n^2),$$

for $n = \{1,2\}$ and $t = 1,\dots,T$, we can derive the first moment of the log of process (12) using the technique described in Ullah (2004, Section 2.2). In particular, we use his Lemma 1, that we report blow for the reader's convenience.

Lemma 1. *Let $h : \mathbb{R}^u \mapsto \mathbb{R}^v$ be an analytic function of the normal random vector $\boldsymbol{y}$ with mean vector $\boldsymbol{m}$ and covariance matrix $\boldsymbol{S}$ such that $\mathbb{E}h(\boldsymbol{y})$ exists, then*

$$\mathbb{E}h(\boldsymbol{y}) = h(\mathbb{D}) \cdot \mathbf{1},$$

where $\mathbb{D}$ is the derivative operator $\mathbb{D} = \boldsymbol{m} + \boldsymbol{S}(\partial/\partial \boldsymbol{m})$.

Proof. Ullah (2004, Lemma 1). □

Notice that the k-th power of the operator $\mathbb{D}$ that will be used throughout the section must be applied as $\mathbb{D}^{k-1}(\mathbb{D}\cdot\mathbf{1})$ and not as $(\mathbb{D}^{k-1}\mathbb{D})\cdot\mathbf{1}$. Furthermore, even when $\boldsymbol{\mu} = \mathbf{0}$ the operator has to be applied with the mean symbol $\boldsymbol{\mu}$ which will be put to zero at the end of the computations. For more details and examples refer to Ullah (2004).

Now, taking the log of (12) we obtain

$$\beta_1 + \eta_{1,t} + \log\Big(1 + \alpha \exp(\beta + \mu_t + \eta_t)\Big) - \log\Big(1 + \alpha \exp(\mu_t)\Big). \tag{13}$$

In order to study if the expectation of this expression is time invariant or not, we need to compute the expectations of the last two addends, which are nonlinear functions of two normal random quantities:

$$(\beta + \mu_t + \eta_t) \sim \mathrm{N}\Big(\beta + \delta t, \sigma^2 t + \tau^2\Big), \qquad \text{and} \quad \mu_t \sim \mathrm{N}\Big(\delta t, \sigma^2 t\Big),$$

with $\delta = \delta_2 - \delta_1$, $\sigma^2 = \sigma_1^2 + \sigma_2^2$ and $\tau^2 = \tau_1^2 + \tau_2^2$. Now, since $\log(1 + \alpha \exp(x))$ is analytic we can expand it around zero as

$$\log(1 + \alpha \exp(\mathbb{D})) = \log(1 + \alpha) + \sum_{i=1}^{\infty} c_i \mathbb{D}^i, \tag{14}$$

where the expansion coefficients c_i depend only on α. Table 1 reports the first five quantities c_i and $\mathbb{D}^i \cdot 1$ necessary to compute the expectation of $\log(1 + \alpha \exp(y))$ for a random variable y with mean m and variance s^2.

So, in order to apply Lemma 1 to the third and fourth addend of (13), we should notice that the function h is identical in both cases; what changes are only the mean

i	c_i	$\mathbb{D}^i \cdot 1$
0	$\log(1+\alpha)$	1
1	$\frac{\alpha}{1+\alpha}$	m
2	$\frac{\alpha}{2(1+\alpha)^2}$	$m+s^2$
3	$\frac{(\alpha-\alpha^2)}{6(1+\alpha)^3}$	$m^2+ms^2+s^2$
4	$\frac{(\alpha-4\alpha^2+\alpha^3)}{24(1+\alpha)^4}$	$m^3+m^2s^2+3ms^2+s^4$

Table 1 First five quantities for the expansion of $\mathbb{E}\big[\log\big(1+\alpha\exp(y)\big)\big]$ with y normal random variable with mean m and variance s^2.

and variance values of the two random variables:

$$\mathbb{E}\Big[\log\Big(1+\alpha\exp(\beta+\mu_t+\eta_t)\Big)-\log\Big(1+\alpha\exp(\mu_t)\Big)\Big]=$$
$$=\log\Big(1+\alpha\exp(\mathbb{D}_1)\Big)-\log\Big(1+\alpha\exp(\mathbb{D}_2)\Big)=\sum_{i=1}^{\infty}c_i(\mathbb{D}_1-\mathbb{D}_2)\cdot 1,$$

where $\mathbb{D}_1$ and $\mathbb{D}_2$ are the derivative operators for the two normal variables, respectively.

Limiting the computations to the first four terms of the expansion we get the results in Table 2. While the first three terms are time-invariant, from the fourth

i	$\mathbb{D}_1^i \cdot 1$	$\mathbb{D}_2^i \cdot 1$	$(\mathbb{D}_1^i \cdot 1)-(\mathbb{D}_2^i \cdot 1)$
0	$\log(1+\alpha)$	$\log(1+\alpha)$	0
1	$\beta+\delta t$	δt	β
2	$\beta+\delta t+\sigma^2 t+\tau^2$	$\delta t+\sigma^2 t$	$\beta+\tau^2$
3	$(\beta+\delta t)^2+(\beta+\delta t+1)(\sigma^2 t+\tau^2)$	$\delta^2 t^2+\delta\sigma^2 t^2+\sigma^2 t$	$\beta^2+\beta\tau^2+\tau^2+(2\beta\delta+\beta\sigma^2+\delta\tau^2)t$

Table 2 First four terms of the expansions (14) for the last two addends of equation (13).

term on it is clear that the expectation of (13) does depend on time. Therefore that process and its exponential (12) are not stationary. We can conclude that indexes of form (10) are not stability preserving, even though they may appear so since the nonstationary components of the mean are multiplied by coefficients that are small in absolute value. In Table 3 the coefficients of orders $i=1,2,3,4$ are computed for few values of $\alpha=\alpha_2/\alpha_1$: notice that the sequence is the same for $\alpha=x$ and $\alpha=1/x$.

c_i	$\alpha=10^{\pm 1}$	$\alpha=2^{\pm 1}$	$\alpha=1$
c_1	0.0909	0.3333	0.5000
c_2	0.0413	0.1111	0.1250
c_3	0.0113	0.0123	0.0000
c_4	0.0017	-0.0031	-0.0052

Table 3 Coefficients of the expansion (14) for different values of α.

We proved that the mean of the process (11) is not time invariant when log-prices are normal, but similar results could be obtained under more general distributional assumptions (cf. Ullah, 2004, Section 2.3).

5 A simulation experiment

The experiment is based on simulated prices for $N = 3$ commodities, $K = 3$ sites and $T = 100$ time points. The prices have been generated using the exponential of Gaussian I(1) processes and are cointegrated across only two countries, while prices in the third country follow idiosyncratic I(1) processes. The simulated prices are depicted in the first column of Figure 3 and the elementary price ratios for each non-redundant pair of sites are shown in the second column of the same figure. From this graph it is clear that a well-behaved system of indexes should reveal that the prices of places 2 and 3 have stable ratios, while the prices of place 1 tend to diverge with respect to the other two sites' prices.

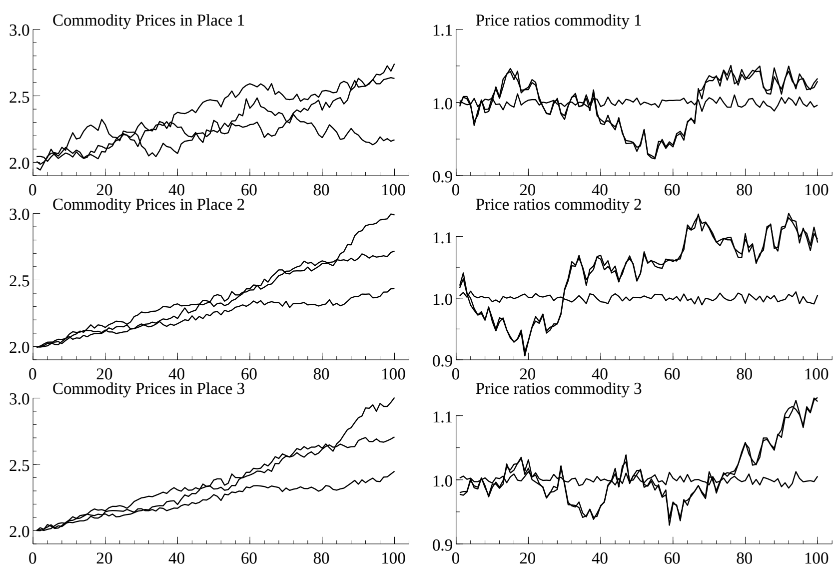

Fig. 3 *First column*: time series of simulated prices for 3 commodities in 3 sites. The first site prices are not cointegrated with the prices of the same commodity in the other sites, while prices in the second and third site have cointegrated prices for the same commodity. *Second column*: time series of price ratios across space for each commodity.

The quantities were obtained from the exponential of nine independent random walks with Student's t increments. The choice of Student's distribution is based on the consideration that in a supermarket, due to promotions, advertisement and openings of neighbouring competitors, quantities should move more erratically than prices. However, the choice of the distribution of the increments does not affect the conclusions of the experiment[11].

The index formulas applied to the simulated data are the following. Let S be the number of the situations to be compared and denote with $P_{a,b}$ the generic index for comparing the situation b with situation a (base).

GEKS (Gini, 1924; Eltetö and Köves, 1964; Szulc, 1964)]
$P_{a,b} = \prod_{s=1}^{S} (F_{a,s} \cdot F_{s,b})^{1/s}$, where $F_{a,b}$ is a Fisher index with base a.

UCW (Gerardi, 1982)
$P_{a,b} = (\sum_n p_{n,b} \cdot q_n)/(\sum_n p_{n,a} \cdot q_n)$ with $q_n = \prod_s q_{n,s}^{1/s}$.

ECLAC (ECLAC, 1978)
$P_{a,b} = (\sum_n p_{n,b} \cdot q_n)/(\sum_n p_{n,a} \cdot q_n)$ with $q_n = \sum_s q_{n,s}$.

GK (Geary, 1958; Khamis, 1972)
$P_{a,b} = P_b/P_a$ with

$$\begin{cases} P_s = (\sum_n p_{n,s} \cdot q_{n,s}) \,/\, (\sum_n \pi_n \cdot q_{n,s}) \\ \pi_n = \left(\sum_{s=1}^{S} p_{n,s} \cdot q_{n,s}/P_s\right) \,/\, \left(\sum_{s=1}^{S} q_{n,s}\right). \end{cases}$$

WCPD (Summers, 1973; Rao, 2005)
$P_{a,b} = \exp(\pi_b - \pi_a)$ where $\pi_1 = 0$ and for $s = 2, \ldots, S$ the π_s are those which solve the weighted least squares problem

$$\min_{\boldsymbol{\pi},\boldsymbol{\beta}} \sum_{r=2}^{S} \sum_{m=1}^{N} w_{m,r} \left[\log p_{m,r} - \sum_{s=2}^{S} \pi_s D_{m,r}^{(s)} - \sum_{n=1} \beta_n E_{m,r}^{(n)} \right]^2,$$

with $w_{m,r} = (p_{m,r} \cdot q_{m,r})/(\sum_{m=1}^{N} p_{m,r} \cdot q_{m,r})$ expenditure share of product m in country r, and

$$D_{m,r}^{(s)} = \begin{cases} 1 & \text{for } r = s, \\ 0 & \text{otherwise} \end{cases} \qquad E_{m,r}^{(n)} = \begin{cases} 1 & \text{for } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

As proved by Rao (2005), this system is equivalent to the one proposed by Rao (1990), defined by $P_{a,b} = P_b/P_a$, where the P_s's solve the system

$$\begin{cases} P_s = \prod_{n=1}^{N} \left(\frac{p_{n,s}}{\pi_n}\right)^{w_{n,s}} \\ \pi_s = \prod_{s=1}^{S} \left(\frac{p_{n,s}}{P_s}\right)^{w_{ij}^*}, \end{cases}$$

[11] For brevity's sake, the quantities and the exact processes' formulas are not reported in the paper, but the software developed for carrying out all the simulations is available from the author on request. The code is written in Ox, so the reader interested in running the software should download the Ox interpreter from www.doornik.com.

with $w^*_{n,s} = w_{n,s} / \sum_{s=1}^{S} w_{n,s}$.

MST (Martini, 1992; Zavanella, 1996; Hill, 1999)]
Let $\boldsymbol{D} = \{d_{i,j}\}_{i,j\in\{1,\dots,S\}}$ a matrix of (symmetric) measures of similarity between situation i and situation j, based on either prices or quantities, or on both; and let $G_{\boldsymbol{D}}$ be the minimum spanning tree graph derived from matrix $\boldsymbol{D}$. Then, denoting with $F_{a,b}$ the Fisher price index with base a,

$$P_{a,b} = \begin{cases} F_{a,b} & \text{when } a \text{ and } b \text{ are directly linked} \\ P_{a,i_1} \cdot P_{i_1,i_2} \cdot \ldots \cdot P_{i_M,b} & \text{otherwise,} \end{cases}$$

where $\{(a,i_1),(i_1,i_2),\dots,(i_M,b)\}$ is any set of vertices with edges linking a with b. There are several ways of measuring the closeness (usually to proportionality) of any two situations. Since the main concern of this work is on price indexes, then the following similarity measure based on price vectors proposed by Diewert (2002) has been implemented:

$$d_{i,j} = \sum_{n=1}^{N} \left[\left(\frac{w_{n,i} + w_{n,j}}{2} \right) \left(F_{i,j}^{-1} \frac{p_{n,j}}{p_{n,i}} + F_{i,j} \frac{p_{n,i}}{p_{n,j}} - 2 \right) \right]$$

where $w_{n,s}$ is the expenditure share of commodity n in situation s defined above.

This set of transitive multilateral systems has been applied in three configurations:

Ensemble the systems have been applied directly to all space and time situations;

Time-based the multilateral systems have been applied only at time $t = 1$, then chained time indexes have been computed for each space situation and space indexes have been derived indirectly;

Space-based for every time point t the systems have been applied with respect to space and the time indexes for each space situation have been derived using a "world" price index chained across time.

5.1 Ensemble indexes

In this configuration all combinations of space and time points are simply treated as situations without differentiating the treatment of the two dimensions. All the properties proposed by Hill (2004) are generally violated and the temporal displacement is maximum $(T-1)$. MST methods could be forced to satisfy temporal fixity and space fixity by joining new space or time situations to the closest situations already in the graph. This would amount to the loss of optimality (minimality), but in general the approximation will be acceptable, especially when new time points are added.

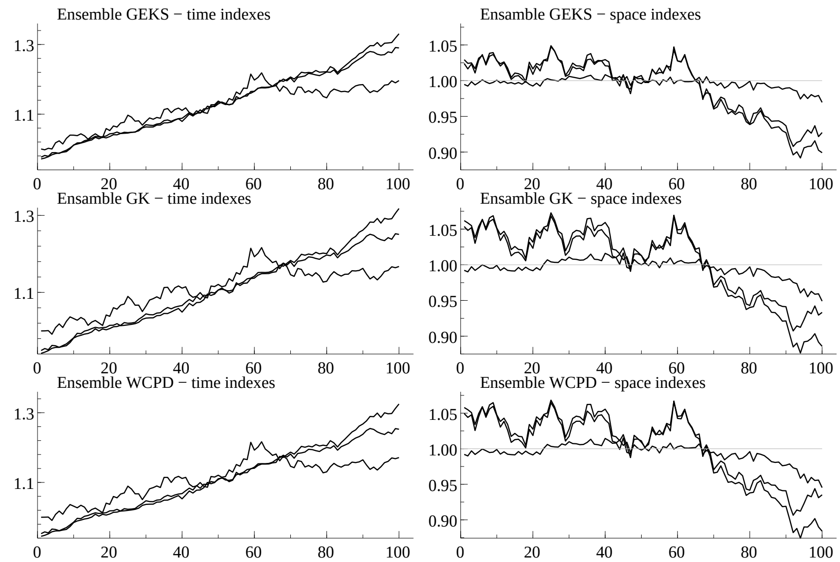

Fig. 4 Ensemble indexes not preserving cointegration.

In fact, it is reasonable to believe the prices in the present are closer to prices in the near past.

As far as cointegration-preservation is concerned, the property should be satisfied only by the UCLAC and the UCW systems, which are in the form specified in equation (9). Indeed, none of the other index formulas is either in the form (8) or (9). If an opportune discrepancy function is chosen, the MST could be forced to chose the paths that minimize the divergence of the index numbers. At the moment, no such function has been studied, but I expect the one used here (Diewert, 2002) to do a reasonable job, being price proportionality (identity) related to a particular form of cointegration-preservation.

Now, from Figure 3 it is clear how the prices of one of the three sites tend to diverge form those in the other two places starting from around time $t = 65$. This is probably a consequence of the choice of generating random walks with correlated increments also for the non-cointegrated prices. The choice was based on the consideration that it is very unlikely that the same commodity's price increments are uncorrelated across space. The result is that the divergence starts to become evident after a while.

So, what a good system of price indexes should indicate is that two places share almost the same prices (indexes close to one) and one place has prices that around observation $t = 65$ start diverging.

By observing Figure 4 it appears clear how according to the GEKS, GK and WCDP systems all price indexes tend to diverge (particularly after $t = 65$), erro-

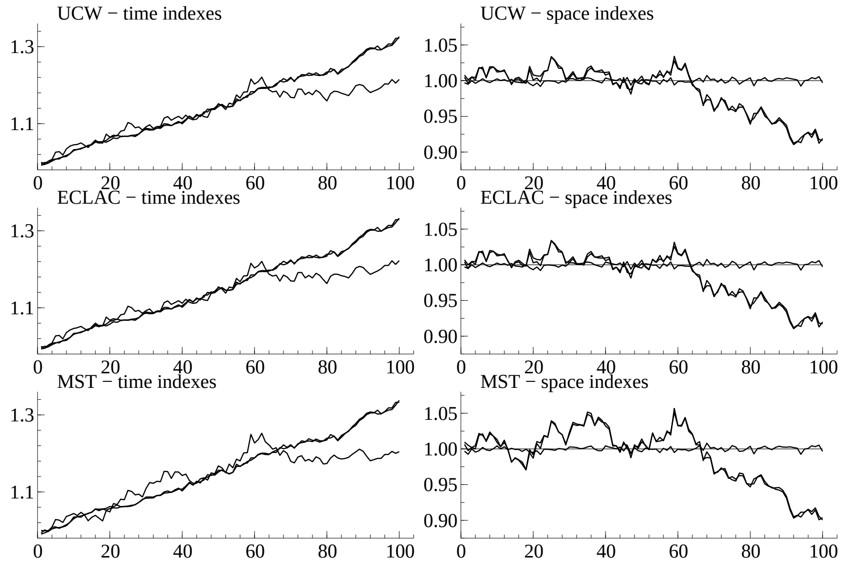

Fig. 5 Ensemble indexes preserving cointegration. The MST system is not cointegration-preserving, but it seems to be a well approximation to such a system.

neously indicating that in all the considered places prices are drifting apart. On the contrary, we know that in two countries the prices of the same commodity are almost identical. There seems to be some kind of *dragging effect* of the indexes computed for site-pairs where prices are actually diverging.

As expected, the indexes series depicted in Figure 5 demonstrate that the UCW and UCLAC systems are preserving the cointegration for the two places where prices are cointegrated, while comparisons with the place with non-cointegrated prices stress their divergence.

The very good behaviour of the MST system is striking, but there are two facts to be taken into account. On the one hand, for two of the three places we simulated prices that are almost proportional (actually almost identical), where proportionality is the target of the discrepancy measure proposed by (Diewert, 2002), on the other hand a better discrepancy measure with cointegration as objective could be designed for those situations where prices are pairwise cointegrated but not almost proportional and this would probably yield a performance similar to the one we are observing here.

5.2 Time-based indexes

Time-based indexes are built as chained Fisher price indexes across time for every single site in sample, initialised at time $t = 1$ with one of the six multilateral systems. Time consistency and time fixity hold by construction, while space fixity depends on the choice of the multilateral system used for initializing the indexes: among the six multilateral systems considered in this work none preserve space fixity, even though the MST graph may be forced to host a new situation losing its optimality only marginally.

As noted before, these indexes are of the type (erroneously) used in many papers that test the purchasing power parity theory and they do not preserve cointegration. The evolution of the time-indexes depends on the choice of the starting system of space-indexes only by a proportionality factor, so in Figure 6 only the GEKS based indexes are shown. The figure confirms what expected: all three bilateral comparisons diverge and this is particularly evident starting from observation $t = 65$.

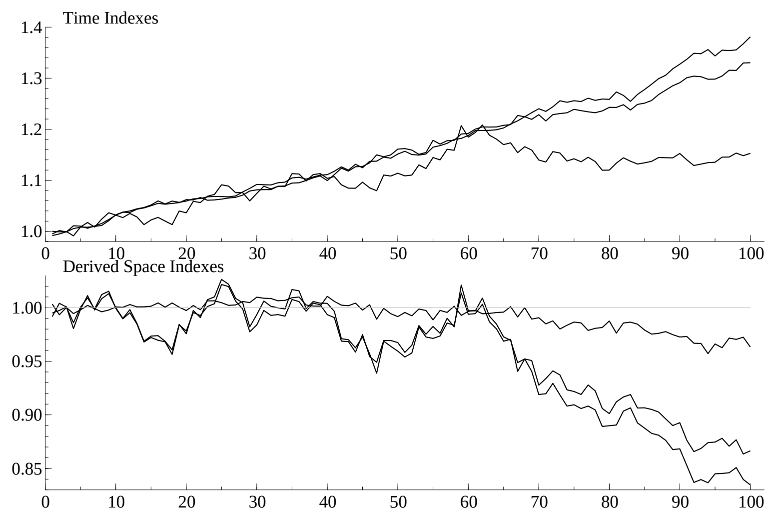

Fig. 6 Time-based indexes with space index at time $t = 1$ being GEKS.

5.3 Space-based indexes

In this configuration, for every t an idiosyncratic system of space-indexes has been computed and the time dimension has been obtained by multiplying the space-indexes by a "world" time-index built applying chained Fisher indexes to all the prices and quantities in the sample.

Without a formal definition of persistence, it is hard to say if the results of this computation preserves cointegration or stability. In fact, since the weights used for the price indexes change with time, the use of the concept of stationarity would certainly lead to a negative answer, while working with less strong concept of some form of weak dependence (e.g. mixing or Granger's (1995) short memory in mean) may suggest conditions under which stability preservation may hold. Figure 7 presents strong evidence in support of the cointegration preservation property. We report only the space-based GEKS system, being the other systems very similar as far as the cointegration preserving behaviour is concerned.

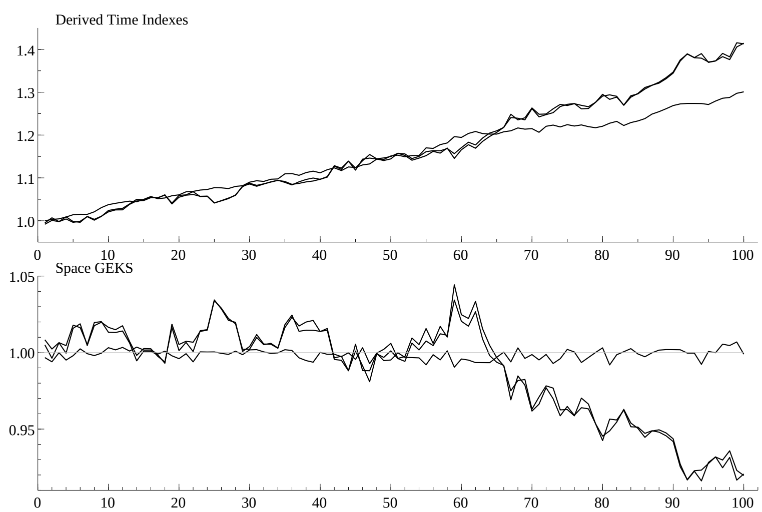

Fig. 7 Space-based EKS indexes updated across time using chained "world" Fisher indexes.

6 Application to real data

We applied the same scheme of the above simulation experiment to real data concerning prices and values of goods sold in two supermarkets and two hypermarkets placed in different areas of Milan. The time series represent total monthly sold values and quantities dating from January 2006 to December 2007. Prices have been recovered by dividing values by quantities, thus, if prices have changed within a month the derived prices are quantity-weighted means. In order to avoid adding a further problem (i.e. the substitution bias) to those considered in this paper, only goods present in the $K = 4$ stores at each of the $T = 24$ months have been used. The total number of goods entering all the indexes is $N = 913$.

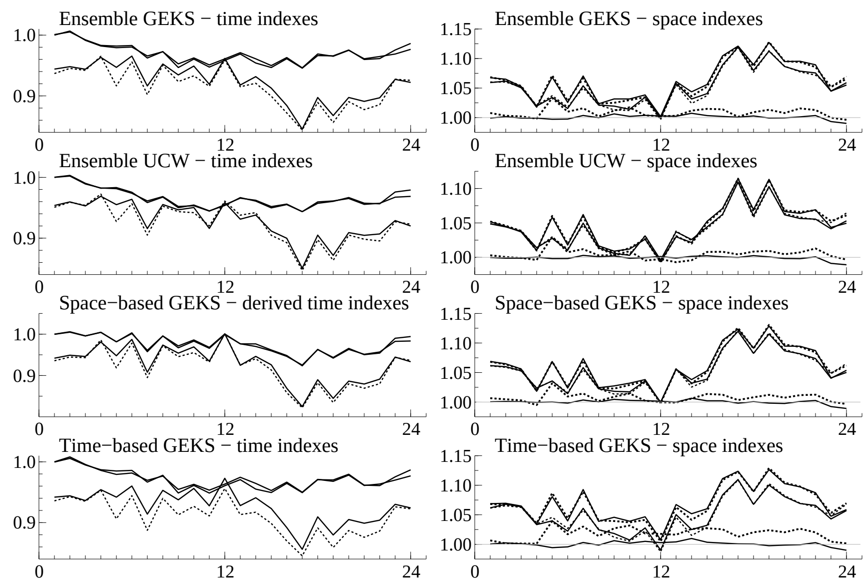

Fig. 8 A selection of space-time indexes for four department stores in Milan.

Figure 8 shows a selection of the multilateral systems computed on these data. Recall that now there are four time series of price indexes across time and $K(K-1)/2 = 6$ non redundant comparisons across space. The first thing that can be noticed is that there are two pairs of stores with similar prices. Indeed, the two supermarkets and the two hypermarkets tend to have close prices, while supermarkets tend to be more expensive than hypermarkets (up to some 10%).

In the considered 24 months the price excursions were modest, so it is hard to notice much from the plots. Nonetheless, the time-based GEKS reveals a divergence of price levels for the two supermarkets that does not show up in the two systems that are expected to be cointegration-preserving (ensemble UCW and space-based GEKS). A much smaller dragging effect can be noted also in the two panels with ensemble GEKS indexes.

These results confirm that time-based indexes represent a bad choice even though the dragging effect for these data was not too strong. The best choices seem to be the ensemble UCW (as well as ensemble ECLAC not shown here) and space-based indexes (the ones not reported had a very similar behaviour). It will be interesting to repeat this application when four or more years of time series become available.

7 Conclusion

The issue of the construction of coherent price indexes across space and time has been somewhat neglected in the scientific literature. A likely motivation for this was the scarce availability, at least in the past, of sufficiently comparable data across both dimensions. These limitations have been partially overcome by scanner data from large-scale retailers.

With this work I propose a stability-preservation property that space-time indexes should satisfy in order to avoid misleading conclusions. In its seminal work on this theme Hill (2004) considers many relevant tests, but the omission of the property proposed here leads him to endorse systems of indexes where the importance of the time dimension prevails on the space dimension. On the contrary, our evidence suggests that space-based indexes should form the basis for well-behaving system of space-time indexes. Nonetheless, Hill proposes a periodic correction of time-based indexes by means of space indexes as well. This surely limits the possible divergence of indexes that should not diverge, but at the cost of introducing a spurious periodicity into the index time series.

Our theoretical and simulation results confirmed by the empirical application should foster further research towards the construction of a complete theory of cointegration-preserving indexes, that should overcome the intrinsic linearity of the definitions of integration and cointegration, possibly extending the work of Ermini and Granger (1993) and Granger (1995) or borrowing from the theory of mixing processes, as I did in this present work.

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