Capital Accounting for Rapidly Obsolescing One-Hoss Shay Individuals in Geometric Cohorts

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At least since Whelan (2002), discussion of computers' impact on productivity growth has emphasized obsolescence as driving high-tech assets' depreciation.[1] Apart from obsolescence, though, service-flows of computers and similar solid-state assets are widely thought to follow the one-hoss shay or (incandescent) "light bulb" pattern. Reconciling one-hoss shay deterioration with obsolescence, and both processes with presumed geometric depreciation of the aggregate computer stock, is one of the hard problems of economic measurement. Recently Diewert and Wei (2015) reconciled cohorts of *simultaneously* one-hoss shay individuals to geometric stock accounting under "somewhat idealized conditions (steady growth in asset investments, steady rates of growth in constant quality prices and constant nominal costs of capital)" — assumptions that worked pretty well in their Australian computer time-series.[2] Diewert and Wei also extended their reconciliation to nearly *arbitrary* cohorts (thus allowing distributed service lives), but they did not explicitly say how much a change in the obsolescence rate would transmit to a change in the aggregate depreciation rate. One may also wonder if *well behaved* cohorts would permit cohort-to-geometric-stock aggregation à la Diewert and Wei even if their somewhat idealized conditions aren't satisfied.

This essay constructs *very* well behaved cohorts — geometric ones (mostly) — from one-hoss shay individuals subject to constant-rate obsolescence. As Diewert and Wei show, cohorts that depreciate at a constant geometric rate aren't necessary for geometric stock accounting, but they are sufficient. Maybe some blend of not-quite geometric cohorts, in "idealized conditions" not quite met, would be adequate for geometric treatment of many asset stocks, without special pleading.

Jointly analyzing geometric cohorts and one-hoss shay individuals requires variational methods to uncover the probability density function of service-lives that reconciles them. Depending on specifics that the essay spells out, that function has a closed-form representation in terms of tangible economic parameters such as the finance rate, new-asset revaluation rate, the rate of quality improvement of new assets over old, and the cohort depreciation rate *net* of realized obsolescence. Under the plausible assumption that the service-life density is slow-to-move, the net depreciation parameter *adjusts* to offset changes in the other parameters, confirming suspicions that an increase in the rate of obsolescence does not transmit one-for-one to the *overall* rate of depreciation: so there is some "clawback."

The assumption of a nearly fixed service-life density is testable, for knowing the density's algebraic form permits fitting age-price regressions with the exact Hulten-Wykoff survivorship correction built in (in simple cases). Moreover, age-price regressions and statistical survivorship studies, which draw on very different data, would identify overlapping subsets of the parameters that inform an asset-type's cohort-level depreciation; if they agree, the system is sound.

[0] Views expressed here are all mine, not my employer's.
[1] Karl Whelan (2002), "Computers, Obsolescence, and Productivity," *The Review of Economics and Statistics,* vol. 84, no. 3 (August): 445–461.
[2] W. Erwin Diewert and Hui Wei (January 16, 2015), "Getting Rental Prices Right for Computers," for IARIW 2015.

The basic mechanism for obsolescence clawback is the contribution of an expected persistent increase in the rate of quality improvement of ever-newer vintages to raising the effective discount rate. The higher discount rate reshapes individual-level age-price profiles, pushing them higher, more toward their age-efficiency counterparts (where the discount-rate is infinite). This partly offsets the direct impact of new cohorts' quality improvements, which depress old cohorts' prices.

Discussions of obsolescence often make much of old assets' *premature* retirements in the face of fitter entrants. The story outlined so far cannot accommodate early quits without a floor price for used-asset sales (below which individuals are scrapped); so one is introduced, which has several effects. First, "service lives" bifurcate into "retirement ages" *versus* (and less than) "lifespans," which have different but related probability distributions: the retirement-age density is observable and amenable to survival analysis, but density of unobserved lifespans is almost invariant, enabling analytical tradeoffs between the net depreciation rate and the density's other arguments (such as obsolescence). Second, the floor price *by itself* alters the depreciation rate only slightly, yet by cutting the tails off future service flows, it also has effects that resemble discounting. These are idiosyncratic: the extra discounting is stronger for short-lived individuals than for long-lived ones. Third, the floor price breaks the heretofore simple duality between geometric cohort age-price profiles and cohort age-efficiency profiles that do not decay at a constant rate. Fortunately, the latter may be approximated as constant-weighted sums of a few geometric processes, which are easy to handle at the cohort and stock levels.

The essay has seven sections. The first reviews the basic individual one-hoss shay model, which the second embeds in a geometric cohort. The third section is the core: it adds obsolescence-as-quality-change to the mix, derives a different service-life density to the no-obsolescence case, and adjusts the depreciation rate to minimize the (Kullback-Leibler) discrepancy between the two; it also adapts for illustration a few rates Diewert and Wei borrowed from the Australian Bureau of Statistics. The section takes pains to distinguish obsolescence from "net" depreciation. The former hinges on comparisons of "new" asset prices (i.e., an old asset's price back when it was new *versus* an actual new asset's price now *versus* the price an old asset would fetch now *were it sold new*); by the Law of One Price across new individuals in the same cohort, obsolescence is thus a cohort-to-cohort comparison. The latter makes more sense as a comparison across individuals within a cohort, as it is driven by different lifespans.

A fragmentary fourth section reminds that discounting is key to clawback: straightline individual age-price forms, which have no discounting, allow full obsolescence pass-through into cohort-level overall depreciation. The fifth section drops obsolescence to revisit the second, but adds a floor price, which compels separate consideration of the retirement and lifespan densities. A difficult sixth section then reintroduces obsolescence, combining the results of the third and fifth sections. This part of the paper is least ready for implementation. Numeric sums-of-geometric representations are given as approximations to nongeometric cohort age-efficiency profiles in both sections 5 and 6. A seventh section concludes.

The paper is mathy, but this is put to the service of the arguments presented. Readers may want *Mathematica* or *Maple* on hand to verify the results.

1. The Individual Case, without Inflation, Obsolescence, or a Scrap Value

Relative to its own service-flows when it was new, the service-flows at age *s* of an individual characterized by a one-hoss shay age-efficiency profile are:

$$\phi(\mathrm{s,L}) = \begin{cases} 1 \; for\ s\ between\ 0\ and\ L \\ 0 \; for\ s \geq L \end{cases} \tag{1}$$

...where *L* is the individual's *correctly anticipated* service-life. Service-flows of a one-hoss shay asset give no hint of when the thing will break, so the assumption of correct anticipation of the service-life is just wrong. We'll replace the assumption soon by an expectational framework; but first consider how to price the individual in (1) relative to an assumed-constant new price. The individual's *resale*-price profile is:

$$\theta(\mathrm{s,L}) = \frac{\int_s^L e^{-r(u-s)}\phi(u,L)du}{\int_0^L e^{-r(u-0)}\phi(u,L)du} = \frac{1-e^{r(s-L)}}{r} \Big/ \frac{1-e^{-rL}}{r} = \begin{cases} \frac{e^{rs}-e^{rL}}{1-e^{rL}} \; for\ s\ between\ 0\ and\ L \\ 0 \quad\; for\ s \geq L \end{cases} \tag{2}$$

...where $r > 0$ is the own rate of return (although if the new price indeed holds constant, this is just *i*, the nominal rate of return), which we'll take to be constant through future horizons. The profile is the ratio of two integrals: the denominator integral is a normalizing constant, so that $\theta(0,\mathrm{L})=1$. The resale price of a used individual is the resale-price profile times the individual's new-supply price: $q(\mathrm{s,L}) = q_0 \times \theta(\mathrm{s,L})$. The denominator integral's *reciprocal*, times that same new-supply price, is also the individual's baseline user-cost:

$$q_0 \frac{r}{1-e^{-rL}} = q_0 \left(r - \partial \frac{e^{rs}-e^{rL}}{1-e^{rL}} \Big/ \partial s \right) \qquad \ldots^{3} \tag{3}$$

Away from age zero, the user-cost gives way to a *rental*-price profile, which always decomposes into the product of the user-cost and the age-efficiency profile:

$$q_0\left(r\,\theta(\mathrm{s,L}) - \frac{\partial\theta(\mathrm{s,L})}{\partial s}\right) = q_0\left(r - \frac{\partial\theta(0,\mathrm{L})}{\partial s}\right) \times \phi(\mathrm{s,L}) \tag{4}$$

...where, by abuse of notation, $\partial\theta(0,\mathrm{L})/\partial s$ means the evaluation at age zero of the age-derivative of the resale-price profile, as in (3). Form (4) is the key to constructing *productive* stocks of capital: as a ϕ-weighted average of past real investment flows (which therefore are now of various ages), all priced by the user-cost. Alternatively, solving form (4) for ϕ shows the age-efficiency profile to be the ratio of the age-*s* rental-price to the user-cost — a useful relationship that holds at any level of aggregation. For one-hoss shay individuals, of course, the decomposition doesn't mean much, since $\phi=1$ for $s<L$.

[3] Evaluate the age-derivative at *s*=0. The result, times the indicated "–" sign, is the baseline depreciation rate of a new one-hoss shay asset: $r/(e^{rL}-1)$. *Baseline*, because obsolescence is not considered yet. Note that form (3) would seem to lack the usual revaluation term: $-\partial q_0/\partial t$. But if we take seriously that *r* is an *own* rate of return — i.e., that $r = i - \partial ln q_0/\partial t$, where *i* is a nominal finance rate — then the revaluation term is indeed present implicitly. It's just that $\partial ln q_0/\partial t = 0$, so far. Note also that *r* needn't be constant across calendar time, though my derivations do hold *r* fixed through the future horizons considered in integrals that are reevaluated at each calendar instant.

2. The Averaged/Cohort Case, without Inflation, Obsolescence, or a Scrap Value

The presence of that *other* argument in form (2) — not just *s*, but *L* too — as well as the implausibility of correctly anticipating a one-hoss shay individual's service life, compels consideration of probabilistic aggregation across a cohort's possible individuals. To embed one-hoss shay individuals within a geometric cohort — alternatively, to spread an individual's bets as far as a geometric outcome — find the service-life density $f(L)$ that solves:

$$e^{-\delta s} = \int_s^\infty \theta(s,L)\, f(L)dL \tag{5}$$

...where $\delta > 0$ is the constant geometric depreciation rate of the cohort.[4] When $\theta(s,L)$ is of form (2), the first and second variations of (5) with respect to age are:

$$-\delta\, e^{-\delta s} = \int_s^\infty \frac{r\, e^{rs}}{1-e^{rL}}\, f(L)dL \qquad \text{and:} \qquad \delta^2\, e^{-\delta s} = -\frac{r\, e^{rs}}{1-e^{rs}} f(s) + \int_s^\infty \frac{r^2\, e^{rs}}{1-e^{rL}}\, f(L)dL \tag{6}$$

Substituting the first variation into the second, solving for $f(s)$, then imposing the lower limit of integration $s \rightarrow L$, yields:

$$f(L) = e^{-\delta L}(1 - e^{-rL})\delta(\delta+r)/r \tag{7}$$

As the difference between two exponentials, the density resembles a Gamma. It integrates over all nonnegative *L* to unity and has mean $1/\delta + 1/(r+\delta)$ and variance $1/\delta^2 + 1/(r+\delta)^2$. Moreover, $f(L)$ implies a survival function:

$$1 - F(L) = e^{-\delta L}[1+\delta(1 - e^{-rL})/r] \tag{8}$$

...that is relevant for the Hulten-Wykoff correction to survivors' bias in age-price regressions. Rather than using an off-the-shelf survival distribution (e.g. one of Winfrey's) to correct the fit of a thoughtless regression:

$$ln(q_{is}/q_{i0}) = \beta\, s + \varepsilon_i \qquad i=1,\ldots,n \tag{9a}$$

...where q_{is}/q_{i0} is the ratio of an individual's age-*s* resale price to its original price,[5] ε_i is an everyday error term, and $-\beta$ is taken for δ — instead impose *ex ante* the correction that is implied by the assumed individual-level one-hoss shay form[6]:

$$ln(q_{is}/q_{i0}) = -ln[1+\delta(1 - e^{-rs})/r] + \varepsilon_i \qquad i=1,\ldots,n \tag{9b}$$

Both δ and *r* are identified. Alternatively, and as a check on the empirical consistency of the one-hoss shay / geometric specification, fit (7) or (8) with observations on retirement ages. If the specification is correct, then (7)/(8) and (9b) should yield similar estimates for δ and *r* from quite dissimilar data.

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[4] The as-yet unknown $f(L)$ is defined all the way down to 0, in principle, so why is the integral only over $L \geq s$? From (2), remember that $\theta(s,L)$=0 for $0 \leq L \leq s$, so $\int_0^s 0\, f(L)dL = 0$. The same consideration is at work in (10), below.

[5] This departs from the set-up so far, which took q_0 to be constant across individuals, by the Law of One Price.

[6] That is, subtract the log of the right side of (8), with *s* replacing *L*, from $-\delta s$, the uncorrected right side of (9a). The correction follows from noting that the *survivors'* average resale profile is: $\tilde{\theta}(s) = \int_s^\infty \theta(s,L) f(L|L \geq s)dL$, where $f(L \mid L \geq s) = f(L)/(1-F(s))$, while the proper cohort average resale profile is: $\bar{\theta}(s) = \int_s^\infty \theta(s,L) f(L)dL$, here $e^{-\delta s}$.

Applying the same density (7) to evaluate the expected age-*s* rental-price for one-hoss shay individuals within a geometric cohort gives:

$$q_0\,(r+\delta)\,e^{-\delta s} \;=\; q_0 \int_s^{\infty} \frac{r}{1-e^{-rL}} \times \frac{e^{-\delta L}\left(1-e^{-rL}\right)\delta(\delta+r)}{r}\, dL \tag{10}$$

...where the individual-level rental-price, $q_0\, r/(1-e^{-rL})$, follows from the left side of (4). As the user-cost and age-*s* rental-price are identical for one-hoss shay individuals, the expected (i.e., cohort) user-cost follows almost trivially, differing from the expected rental-price only in the lower limit of integration:

$$q_0\,(r+\delta) \;=\; q_0 \int_0^{\infty} \frac{r}{1-e^{-rL}} \times \frac{e^{-\delta L}\left(1-e^{-rL}\right)\delta(\delta+r)}{r}\, dL \tag{11}$$

The ratio of the cohort rental-price to the cohort user-cost returns the cohort geometric age-efficiency profile: $\Phi(s) = e^{-\delta s}$.

An aside: One sometimes sees the cohort age-efficiency profile built directly from individual efficiency pieces. Constructing a cohort geometric efficiency profile from individual one-hoss shay profiles is deceptively easy:

$$\Phi(s) = e^{-\delta s} = \int_s^{\infty} \delta e^{-\delta L} \times 1\, dL \tag{12}$$

What is this negative-exponential density? It is not the density of service-lives — that's form (7). Rather, it is a normalized *compound* of the service-life density and individual-level user-costs:

$$\delta\, e^{-\delta L} = \frac{\dfrac{r}{1-e^{-rL}} \times \dfrac{e^{-\delta L}\left(1-e^{-rL}\right)\delta(\delta+r)}{r}}{\displaystyle\int_0^{\infty} \frac{r}{1-e^{-rL}} \times \frac{e^{-\delta L}\left(1-e^{-rL}\right)\delta(\delta+r)}{r}\, dL} \tag{13}$$

The notion of a direct density as a compound holds beyond one-hoss shay. Statistical agencies that assemble cohort age-efficiency profiles as simple weighted averages of individual ϕ-profiles (indeed some do) need to consider what their weights really mean.[7]

3. Individual and Cohort Cases, with Constant-Rate Inflation and Obsolescence, but No Scrap Values

This discussion began by assuming that new-asset purchase prices were constant, and that different cohorts were distinguished only by their different installation dates, not by any underlying differences in quality. Suppose instead the current new supply-price increases at a constant rate π, *and* that newer cohorts are simply *better*, at the current or "frontier" new-supply price, than older ones (which must therefore be devalued) — in fact, better than older cohorts were *when they were the frontier*, and better than older cohorts would be currently were they somehow restored to their original

[7] Martin S. Feldstein and Michael Rothschild make the same mistake in "Towards an Economic Theory of Replacement Investment," *Econometrica*, vol. 42, no. 3 (May 1974), pp. 393-424, particularly pp. 403-404.

luster and durability. (This comparison makes more sense for *non-* one-hoss shay individuals, where the effects of wearing out are palpable.) For individuals with equal service-lives, obsolescence shows itself as a reduction in the service-flows of such fully-restored individuals from an older cohort *relative to flows* from individuals from the frontier cohort, for it is always the frontier cohort that sets the terms for the user-cost. If the rate of improvement ($b > 0$) is constant through calendar-time and horizons, then the service-flows of an age-*s* individual characterized by a one-hoss shay age-efficiency profile vis-à-vis itself when it was new, relative to those of a genuinely new one-hoss shay individual at the frontier, are:

$$\phi(s,L) = \begin{cases} e^{-b(t-v)} \times 1 & for\ s\ between\ 0\ and\ L \\ 0 & for\ s \geq L \end{cases} \tag{14}$$

...where *t* is the calendar-date and *v*, the vintage, is the date the age-*s* cohort was originally installed. (Note $t–v=s$.) This is a narrow conception of obsolescence: it excludes quality improvements that lengthen service-lives, and it only counts devaluation relative to the frontier, not of the frontier as such in the wider marketplace of substitutes and complements. Relative to itself, ϕ still follows form (1), and relative to an intermediate vintage issued a fixed length of calendar-time away, $\phi = e^{-b(v_1 - v_2)}$. The individual's *resale*-price profile, relative to its original, date-*v* supply-price, $q_0(v)$, is:

$$\theta(s,t,v,L) = \frac{\int_s^L e^{\pi u}\, e^{-i(u-s)} e^{-bu} du}{\int_0^L e^{\pi u}\, e^{-i(u-0)} e^{-bu} du} = e^{\pi s} e^{-bs} \frac{1-e^{(i-\pi+b)(s-L)}}{i-\pi+b} \Big/ \frac{1-e^{-(i-\pi+b)L}}{i-\pi+b} \tag{15a}$$

$$= \begin{cases} e^{(\pi-b)(t-v)} \dfrac{e^{(i-\pi+b)s} - e^{(i-\pi+b)L}}{1-e^{(i-\pi+b)L}} & for\ s\ between\ 0\ and\ L \\ 0 & for\ s \geq L \end{cases}$$

...where $e^{\pi s} = e^{\pi(t-v)}$ represents the frontier supply-price at date *t*, relative to what the frontier supply-price was at date $v < t$; $e^{-bs} = e^{-b(t-v)}$ is the *as-if-new* price of a vintage-*v* asset were it restored to its original newness and survivability as of date *t*, relative to the actual date-*t* frontier purchase-price; and $e^{-i(u-s)}$ shows discounting by nominal interest rate *i*, constant through future horizons $u \geq s$. In the second row of (15a), I've restored $t–v$ to the leading term to highlight the distinction between accumulations of naive inflation $e^{\pi(t-v)}$ and realized obsolescence $e^{-b(t-v)}$, on the one hand, and expected future obsolescence (i.e., $i–\pi+b$ instead of just $i–\pi$), on the other.[8]

The resale price of a used individual is the supply price that prevailed when the individual, then on the frontier, was bought new, back at date *v* (that is, $q_0(v)$), times the resale-price profile $\theta(s,t,v,L)$:

$$q(s,t,v,L) = q_0(v)\ \theta(s,t,v,L).$$

For large enough π, $\theta(s,t,v,L)$ might not decline monotonically as age and time progress. An alternative representation shunts purchase-price inflation to the purchase price itself, replaces the historical supply price by the current supply price, $q_0(t) = q_0(v)e^{\pi(t-v)}$, and so adjusts the age-price profile to:

[8] With an eye toward empirical application, one might also distinguish the *b* in the realized obsolescence term from the *b* that supplements $i–\pi$. The former might jostle from year to year, with the latter stable.

$$\theta^*(s,t,v,L) = \begin{cases} e^{-b(t-v)}\frac{e^{(i-\pi+b)s}-e^{(i-\pi+b)L}}{1-e^{(i-\pi+b)L}} & for\ s\ between\ 0\ and\ L \\ 0 & for\ s \geq L \end{cases}. \tag{15b}$$

Both $q_0(t)$ and $q_0(v)$ are or were observed, respectively. A third representation builds realized obsolescence into the old asset's *as-if-new-at-date-t* price: $q_0(v,t) = q_0(t)e^{-b(t-v)} = q_0(v)e^{(\pi-b)(t-v)}$. This is not directly observable, but it might be approximated by hedonic methods, where π–b is the supply-price inflation rate of a constant-quality new asset. The corresponding stripped-down age-price profile:

$$\theta^{**}(s,L) = \begin{cases} \frac{e^{(i-\pi+b)s}-e^{(i-\pi+b)L}}{1-e^{(i-\pi+b)L}} & for\ s\ between\ 0\ and\ L \\ 0 & for\ s \geq L \end{cases}. \tag{15c}$$

...is essentially expression (2) again, but with r explicitly replaced by i–π+b.

Modelers may choose $\theta(s,t,v,L)$, $\theta^*(s,t,v,L)$, or $\theta^{**}(s,L)$ to suit, but each specification implies a different apparent discount factor in the rental price. For a vintage-v, age-s individual at date t, the necessary rental-price specifications — which all evaluate numerically the same — are:

$$q_0(v)\left(i\,\theta(s,t,v,L) - \frac{\partial\theta(s,t,v,L)}{\partial s} - \frac{\partial\theta(s,t,v,L)}{\partial t}\right) = q_0(v)\, e^{(\pi-b)(t-v)}\frac{i-\pi+b}{1-e^{-(i-\pi+b)L}} \tag{16a}$$

$$q_0(t)\left((i-\pi)\theta^*(s,t,v,L) - \frac{\partial\theta^*(s,t,v,L)}{\partial s} - \frac{\partial\theta^*(s,t,v,L)}{\partial t}\right) = q_0(t)\, e^{-b(t-v)}\frac{i-\pi+b}{1-e^{-(i-\pi+b)L}} \tag{16b}$$

$$q_0(v,t)\left((i-\pi+b)\theta^{**}(s,L) - \frac{\partial\theta^{**}(s,L)}{\partial s}\right) = q_0(v,t)\frac{i-\pi+b}{1-e^{-(i-\pi+b)L}} \tag{16c}$$

...where the relevant discount factors are shaded. For the frontier vintage (i.e., v=t), both $q_0(v,t)$ and $q_0(v)$ reduce to $q_0(t)$. This gives the user cost:

$$q_0(t)\frac{i-\pi+b}{1-e^{-(i-\pi+b)L}}$$

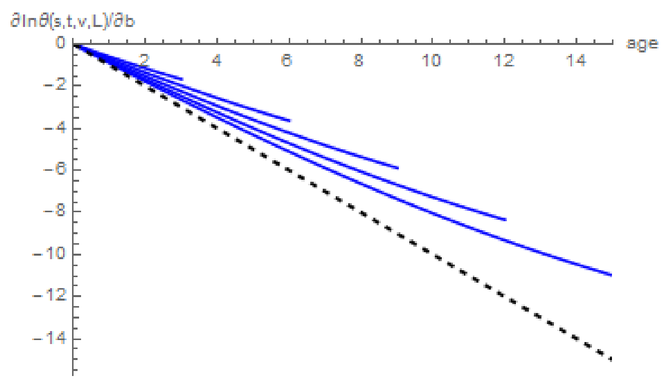
so the ratio of the vintage-v rental-price to the user cost is $e^{-b(t-v)}$, the relative-to-frontier age-efficiency profile of (14). Observe the user cost is the product of the current-supply price and the reciprocal of the denominator integral of (15a).

Note how *different* constant-rate realized obsolescence is as a mode of depreciation, at the individual level, from ordinary "net" channels: wearing out (which is suppressed in one-hoss shay assets) and discounting. The former, as a geometric process, operates equally across short- and long- lived individuals and between the age-efficiency and age-price domains. The latter two vary across individuals, driven by the Law of One Price for new-asset purchases: a cohort's short-lived members must "pay for" their original purchase-value in the form of faster depreciation than long-lived members'. Aggregating to the geometric cohort, where realized obsolescence and net depreciation are summable rates, $b + \delta^* > 0$, effaces the distinction. However, *anticipated* obsolescence does have differential effects, via "pumping up" the effective own-interest rate from $i - \pi$ to $i - \pi + b$. As a higher discount rate elevates the individual age-price profile (in the no-tomorrow limit, all the way to the age-efficiency profile), it slows the pace of individual depreciation; so one should expect, *for an essentially unchanged service-life distribution*, the comparison $b + \delta^o > b + \delta^* > \delta^o > \delta^* > 0$ to prevail for the cohort, where net rate δ^* incorporates the effects of higher discounting, while net rate δ^o does not.

To formalize the argument, calculate the *ln*-change of the resale-price profile (15a)[9] with respect to a marginal increase in the long-run rate of quality improvement:

$$\frac{\partial ln\theta(s,t,v,L)}{\partial b} = -s + \frac{s\, e^{(i-\pi+b)s} - L\, e^{(i-\pi+b)L}}{e^{(i-\pi+b)s} - e^{(i-\pi+b)L}} - \frac{L}{1-e^{-(i-\pi+b)L}} \tag{17}$$

If b never entered the discounting — i.e., if every vintage's improvement came as a complete surprise against prior expectations of no improvement — the only effect would be direct: $\partial ln\ e^{(\pi-b)s}/\partial b = -s$. Instead, individual responses soften the direct effect, depending on age and service-lives. To see the difference, borrow some rates from Diewert and Wei, who supply 6.627% and –14.096% as long-run values of the annual Australian government nominal interest rate and quality-adjusted computer-price growth-rate, respectively. Putting these in continuous terms gives $i = ln(1+.06627) = .06417$ and $\pi-b = ln(1-.14096) = -.15194$. Suppose further that $\pi = .03417$, for a naive own-rate of return of $i-\pi = .03$. This implies $b = .18611$, so quality doubles every 3¾ years — about half the rate of Moore's Law. A plot of (17) with these values, for individuals $L = \{3, 6, 9, 12, 15\}$, shows the departures from the direct effect (though the overall effect is still downward, indicating a hastening of combined depreciation):[10]



Next, inspect resale-price and rental-price profiles for the same five individuals for the distinctly *non*-marginal contrast $b = .18611$ *versus* 0[11] (but keep i=.06417 and π=.03417). In the following six plots, solid blue lines represent the case of rapid quality improvement (b = .18611); dashed red lines show the case of no improvement (b = 0). The top row of three plots are of individual resale-price profiles (15a), (15b), and (15c), left-to-right. The bottom three show the corresponding rental-price profiles (16a), (16b), and (16c), left-to-right, *relative to* their respective supply prices $q_0(v)$, $q_0(t)$, and $q_0(v,t)$. (Only $q_0(v)$ is in fact unchanging as the assets age.) Note all the resale-price profiles begin at 1, per the Law of One Price, while the rental-price profiles begin at levels inverse to their individual service lives. User costs specific to each individual are read off the vertical (i.e., age-0) axes of the rental-price plots; they agree across all three treatments...

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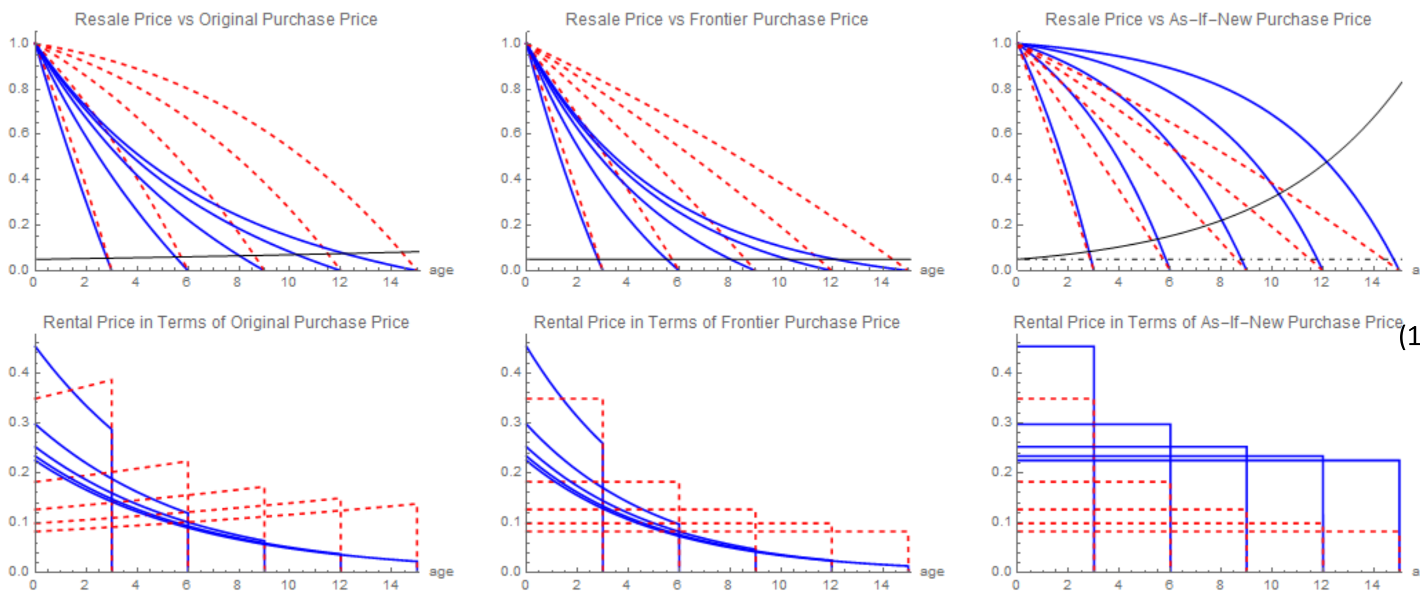
[9] Starting from form (15b) would give the same result, but using (15c) would miss the leading "–s" in (17).
[10] Diewert and Wei work in a discrete-time framework. They also restrict *cohort*-level L to 3 or 4 years.
[11] *N.B.*: Until 1985, BEA's computer price index was flat and did not adjust for quality improvement — i.e., π=b=0.



(18)

Now compare column-by-column: The left two plots show the full effects of inflation, realized obsolescence, and discounting, and so agree with resale and rental profiles (15a) and (16a), with $q_0(v)$ set to 1. The middle two plots abstract from naive inflation, per forms (15b) and (16b), with $q_0(t)$ implicitly held to 1 through time. Blue lines in the middle plots fall a bit more steeply than in the left two, while dotted red lines in the middle plots either fall more steeply than in the left plots or do not increase at all; the differences are all due to the absence of the leading $e^{\pi s}$ from (15b) and (16b). The right two plots remove both inflation and realized obsolescence, $e^{(\pi-b)s}$, as per (15c) and (16c), but fix $q_0(v,t)$ at 1. Here the solid blue lines are everywhere above their dashed red counterparts (which match the dashed red schedules of the middle plots), showing the effects of a large positive b on the discounting.

All six plots further agree that neither realized nor anticipated obsolescence induces early retirements(!) This flies against the received wisdom on computer retirements, and indeed against common sense. Mechanically, early retirements in the plots would require a positive scrap-value / floor for the resale-price, below which an individual would be retired (and its further value in the cohort set to zero). Even small scrap values would have big effects in facilitating early retirements. For example, the thin black lines in the three resale-price plots equivalently represent the profile through time since date v of a scrap price that began at 5 percent of the vintage-v original supply price, then inflated thereafter at the same rate as the naive rate on new asset prices (π = .03417), though without quality improvement. Individual retirements at a positive price would always occur sooner than at the zero-price lifespan, disproportionately so for long-lived individuals; and the effect is amplified under obsolescence.[12]

[12] This is easy to see in the first two resale-price plots, where the longest-lived individual (with L=15 years at a zero floor-price) would be retired at age 14.4 absent obsolescence *versus* age 12.1 for persistent b=.18611. In the third resale-price plot, there are two scrap-price schedules: the solid one *increases* at rate .18611 to maintain a proper comparison with a $q_0(v,t)$ that is artificially fixed at 1 despite rapid quality improvement, while the dot-dashed one is fixed at .05 (as in the middle plot), given no quality improvement. Still, compare the age where the 15-year dashed red line intersects the thin dot-dashed black line (14.4) to the age where the 15-year blue line intersects the thin upwardly-curved black line (12.1).

Nonetheless, standard SNA practice and a dearth of reliable scrap-price data restrict attention, for now, to a zero scrap-value — permitting no obsolescence-induced retirement of any individual asset. Given this restriction, it is hard to see how the *distribution* of service-lives would change much, either, relative to the case of no obsolescence. This is the hook we will use to compare depreciation rates across the two cases. Following the steps of (5)-(7), we seek the service-life density that solves:

$$e^{-(\delta^* + b)s} = \int_s^\infty e^{-bs} \frac{e^{(i-\pi+b)s} - e^{(i-\pi+b)L}}{1 - e^{(i-\pi+b)L}} \, f(L)dL \tag{19}$$

...where δ^* prevails for $b>0$, as against δ^o for $b=0$. (Overall depreciation is the sum: $\delta^* + b$ *versus* δ^o.) The hypothesis that obsolescence is over-counted implies $\delta > \delta^*$, but by how much? Canceling e^{-bs} from both sides and carrying out the same steps as in (5)-(7), find:

$$f(L) = e^{-\delta^* L}\,(1 - e^{-(i-\pi+b)L})\,\delta^*\,(\delta^* + i - \pi + b)/(i - \pi + b) \tag{20}$$

...which has mean $1/\delta^* + 1/(\delta^* + i - \pi + b)$, variance $1/\delta^{*2} + 1/(\delta^* + i - \pi + b)^2$, and survival function:

$$1 - F(L) = e^{-\delta^* L}\,[1 + \delta^*(1 - e^{-(i-\pi+b)L})/(i - \pi + b)] \tag{21}$$

Diewert and Wei cite 39.22% as the Australian Bureau of Statistics' discrete annual depreciation rate for computers, given that agency's use of quality-adjusted deflators, which implies a continuous rate of $ln(1+.39220) = .33089$. Now suppose, with Diewert and Wei, that ABS "got it right," so the continuous depreciation rate is the sum of the rates of realized obsolescence and ordinary depreciation in the presence of anticipated obsolescence: $.33089 = b + \delta^*$. For obsolescence rate $b = .18611$ as above, this implies $\delta^* = .14478$. As an alternative, suppose ABS "got it very wrong" and did not account for quality improvement, like BEA before 1985. This is tantamount to using form (7) for the service-life density, with $r = i - \pi = .03$ as above. Although densities (7) and (20) resemble each other, nonetheless δ cannot be adjusted enough, for an unchanged i–π, to close fully the discrepancy between the two densities that is induced by cutting b from .18611 to 0. But we can try to come close. One way is to choose δ to minimize the Kullback-Leibler discrepancy between (7) and its target density, (20):

$$\min_{\delta^o} \int_0^\infty e^{-\delta^o L}\left(1 - e^{-(i-\pi)L}\right)\delta^o \frac{\delta^o + i - \pi}{i - \pi}\, ln\left[\frac{e^{-\delta^o L}\left(1 - e^{-(i-\pi)L}\right)\delta^o \dfrac{\delta^o + i - \pi}{i - \pi}}{e^{-\delta^* L}\left(1 - e^{-(i-\pi+b)L}\right)\delta^* \dfrac{\delta^* + i - \pi + b}{i - \pi + b}}\right] dL \tag{22}$$

By this criterion, the continuous-time δ^0 that nearly immobilizes the density function is $\delta^0 = .19398$, which is about what one would find just by equating means: $\delta^0 = .19274$. The resulting no-obsolescence density approximately coincides with the density that prevails when $b = .18611$. (See the plot on the next page.) The procedure makes δ an implicit function of the effective own-interest rate: $i - \pi + b$.[13] We finish confirming the suspected ordering:

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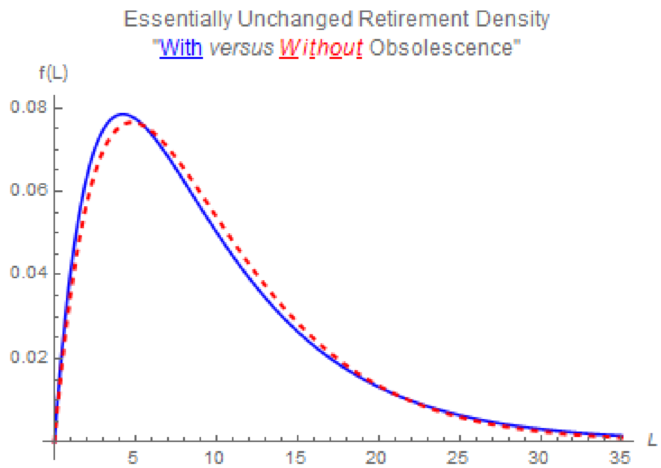
[13] Now, i and π might vary as well, so δ^0 would too. Yet Diewert and Wei take the rates to represent long-run averages, and so will I. Still, grounding a nearly fixed density on possibly flighty parameters begs a big question.

$$b + \delta^o > b + \delta^* > \delta^o > \delta^* > 0$$
$$.380 > .331 > .194 > .145 > 0$$

The difference between δ^o and δ^* is over a quarter the size of *b* and so is an unignorable clawback. If instead ABS "only got it somewhat wrong," then it added *b* to the no-obsolescence δ^o without adjusting for anticipations: $b + \delta^o = .380$.



(23)

Of course, one should verify the presumption of an unchanged service-life distribution by estimating the system's parameters in a consistently corrected age-price regression, either:

$$ln[q_{j,v,s}/q_0(v)] = (\pi - b)s - ln[1+\delta^*(1 - e^{-(i-\pi+b)s})/(i - \pi + b)] + \varepsilon_j^A \qquad j=1,\ldots,n \qquad (24a)$$

or:

$$ln[q_{j,v,s}/q_0(t)] = -b\, s - ln[1+\delta^*(1 - e^{-(i-\pi+b)s})/(i - \pi + b)] + \varepsilon_j^B \qquad j=1,\ldots,n \qquad (24b)$$

The first form (of a vintage-*v*, age-*s* individual's price relative to its original date-*v* price) identifies δ^*, *i*, and $\pi - b$. The second form (of the same vintage-*v*, age-*s* individual's price relative to the current frontier price) identifies δ^*, *b*, and $\pi - i$. A survival study based on (20) or (21) would identify δ^* and $i - \pi + b$. A joint price and survival study would be more convincing than either alone.

Applying density (20) to evaluate the expected/cohort age-*s* rental-price for obsolescing one-hoss shay individuals within a geometric cohort yields:

$$q_0(t)\,(i - \pi + b + \delta^*)\, e^{-\delta^* s - b(t-v)} = q_0(t)\, e^{-b(t-v)} \int_s^\infty \frac{i-\pi+b}{1-e^{-(i-\pi+b)L}} \times \frac{e^{-\delta^* L}\left(1-e^{-(i-\pi+b)L}\right)\delta^*(\delta^* + i - \pi + b)}{i-\pi+b}\, dL \qquad (25)$$

…where the individual-level rental-price, $q_0(t)e^{-b(t-v)}(i - \pi + b)/(1 - e^{-(i-\pi+b)L})$, is from (16b). The expected/ cohort user-cost is nearly the same, but it is evaluated at *s*=0 (and so *v*=*t*):

$$q_0(t)\,(i - \pi + b + \delta^*) = q_0(t) \int_0^\infty \frac{i-\pi+b}{1-e^{-(i-\pi+b)L}} \times \frac{e^{-\delta^* L}\left(1-e^{-(i-\pi+b)L}\right)\delta^*(\delta^* + i - \pi + b)}{i-\pi+b}\, dL \qquad (26)$$

The ratio of the cohort rental-price (25) to the cohort user-cost (26) returns the cohort geometric age-efficiency profile: $\Phi(s) = e^{-(\delta^{*}+b)s}$. As we have seen, this decays more slowly than rate $\delta+b$, due to the obsolescence-boosted discount rate.

4. Fragility of the One-Hoss Shay Result: Interest-Insensitive Individual Age-Price Profiles

Earlier I claimed that the service-life density (7) "resembles a Gamma." In fact, to the nearly coincident densities plotted in (23) may be added a third, a *genuine* Gamma form:

$$f(L) = \frac{\tilde{\delta}^{\alpha} L^{\alpha-1}}{\Gamma(\alpha) e^{\tilde{\delta} L}} \tag{27}$$

...with particular values $\tilde{\delta}$ = .208 and α = 2. Moreover, the Gamma retirement density is exact for an obsolescing individual age-price profile:

$$\tilde{\theta}^{*}(s,t,v,L) = e^{-b(t-v)}(1-s/L)^{\alpha-1} \qquad \ldots^{14} \tag{28}$$

...in the geometric cohort:

$$q_0(t)\, e^{-\tilde{\delta}s-b(t-v)} = q_0(t)\int_s^{\infty} e^{-bs}\left(1-\frac{s}{L}\right)^{\alpha-1} \frac{\tilde{\delta}^{\alpha} L^{\alpha-1}}{\Gamma(\alpha) e^{\tilde{\delta} L}}\, dL \tag{29}$$

On the other hand, density (27) is *also* exact for parameterizations of (28) and (29) with b set to 0 (or to any other value, for that matter). Unlike the one-hoss shay case, form (28) — which no one would apply to a computer — allows full obsolescence pass-through, because it does not depend on the discount rate. Conversely, the implied rental-price (= user-cost $\times$ age-efficiency profile) *is* sensitive to the discount rate, but in a way that is transparent to obsolescence, in that the cohort rental-price:

$$q_0(t)(i-\pi+b+\tilde{\delta})e^{-\tilde{\delta}s-b(t-v)} = q_0(t)\int_s^{\infty}\left(\frac{\alpha-1}{L}+i-\pi+b\right)\times e^{-b(t-v)}\left(1-\frac{s}{L}\right)^{\alpha-2}\frac{(i-\pi+b)(L-s)+\alpha-1}{(i-\pi+b)L+\alpha-1}\frac{\tilde{\delta}^{\alpha}L^{\alpha-1}}{\Gamma(\alpha)e^{\tilde{\delta}L}}\,dL \tag{30a}$$

...divided by the cohort user-cost:

$$q_0(t)(i-\pi+b+\tilde{\delta}) = q_0(t)\int_0^{\infty}\left(\frac{\alpha-1}{L}+i-\pi+b\right)\frac{\tilde{\delta}^{\alpha}L^{\alpha-1}}{\Gamma(\alpha)e^{\tilde{\delta}L}}\,dL \tag{30b}$$

...returns a cohort geometric efficiency profile, $e^{-\tilde{\delta}s-b(t-v)}$, undiminished by obsolescence.

So, just how sure are we that individual computers really are one-hoss shay assets?

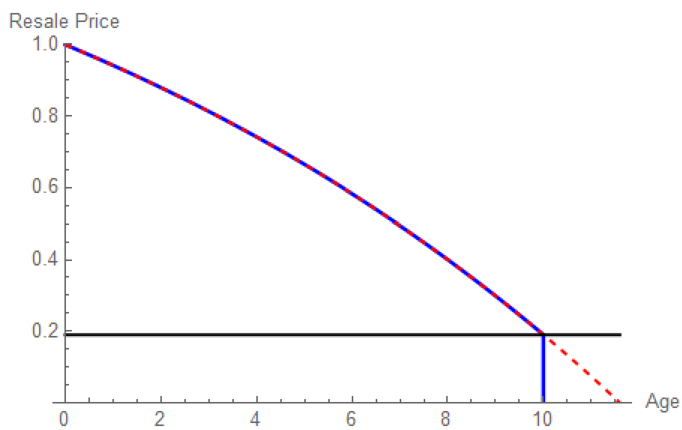
[14] For a sustained discussion of form (32), see Sliker (2014), "Implications of Geometric Cohort Depreciation for Service-Life Distributions," a draft awaiting supervisory approval for admission to BEA's Working Paper collection.

5. Individual and Cohort Cases, with a Positive Scrap Price but No Inflation or Obsolescence

The framework so far cannot induce premature retirements if a one-hoss shay individual remains in service for as long as it still has a shred of value. To prepare for obsolescence-induced retirements, adopt the expedient of a floor-price that is a constant fraction $0 < k < 1$ of the quality-*un*adjusted frontier supply price q_0.[15] To focus thoughts, restrict π=b=0 (so $r = i - \pi + b$ is just i again). The resale price of a one-hoss shay individual works out as:

$$q(s,m) = \begin{cases} q_0\left((1-k)\frac{e^{rs}-e^{rm}}{1-e^{rm}}+k\right) = q_0\frac{e^{rs}-\frac{e^{rm}-k}{1-k}}{1-\frac{e^{rm}-k}{1-k}} & for\ s\ between\ 0\ and\ m \\ 0 & for\ s \geq m \end{cases} \tag{31}$$

...where the floor-price is $q_0 k$ and the retirement age is $m < L$.[16] An example profile, for $q_0 = 1$, k =.19 (way too high but at least visible in the plot), r=.07, and m=10, is the solid blue schedule below. (The dotted red line represents the continuance out to L.)



(32)

The corresponding individual rental-price is:

$$\begin{cases} q_0 r\frac{e^{rm}-k}{e^{rm}-1} & for\ s\ between\ 0\ and\ m \\ 0 & for\ s \geq m \end{cases} \tag{33}$$

This is also the user-cost, so the age-efficiency pattern remains the standard one-hoss shay ϕ=1. The net present value integral of the rental-price:

$$q_0\int_s^m e^{-r(u-s)}\, r\frac{e^{rm}-k}{e^{rm}-1}du \tag{34}$$

...falls short of q(s,m) in (31) by the amount $q_0\, k\, e^{-r(m-s)}$, which is the discounted value of the floor-price.

[15] A fuller treatment would show flexible inputs progressively reallocated from older assets toward newer ones.
[16] In principle, m is observable from that final transaction whereby the firm sells the asset to the scrap dealer. The zero floor-price *lifespan*, L, really not observable now, would be $ln[(e^{r\,m} - k)/(1 - k)]/r$. If k=0 then m=L.

Despite the positive floor-price, one may still embed one-hoss shay individuals in a cohort geometric price profile, with a different retirement-age density. The first and second age-variations of:

$$e^{-\delta s} = \int_{s}^{\infty} \left((1-k)\frac{e^{rs}-e^{rm}}{1-e^{rm}} + k \right) f(m)\,dm \tag{35}$$

...give rise to a linear differential equation (in m):

$$f'(\mathrm{m}) - \frac{r}{k}\frac{e^{rm}-k}{e^{rm}-1} f(\mathrm{m}) = -\delta\,\frac{r+\delta}{k}\,e^{-\delta \mathrm{m}} \tag{36}$$

...that solves as:

$$f(\mathrm{m}) = e^{rm}\,(e^{rm}-1)^{1/k-1}\ \mathrm{Beta}[e^{-rm},\ 1/k+\delta/r,\ 2-1/k]\ \delta(\delta+r)/(rk) \qquad \ldots^{17} \tag{37}$$

Alternatively, using the probability density function transformation method,[18] find the density of (zero-price) *lifespans*, which are not observed, but which would be insensitive to early retirements:

$$f(\mathrm{L}) = e^{rL}(e^{rL}-1)^{1/k-1}\ \mathrm{Beta}[1/\big(k+(1-k)e^{rL}\big),\ 1/k+\delta/r,\ 2-1/k]\ (1-k)^{1/k}\ \delta(\delta+r)/(rk) \tag{38}$$

The two densities are similar but not the same. The lifespan density satisfies:

$$e^{-\delta s} = \int_{ln\left[\frac{e^{rs}-k}{1-k}\right]/r}^{\infty} \frac{e^{rs}-e^{rL}}{1-e^{rL}} f(L)\,dL \tag{39}$$

...where the lower limit of integration represents the value of L for which an s-year old asset attains the floor-price k. Forms (35) and (39) in fact represent equivalent approaches to the same problem, with solutions expressed in terms of observable (i.e., m) *versus* invariant (i.e., L) distributional measures of a cohort's durability.

Unfortunately, I have not found survival functions for (37) or (38) with simple representations. This presents a problem for consistently corrected age-price regressions. While an age-price estimation system here would have only three parameters — δ, r, and k — tempting a grid-search with a numeric-integral replacing the survival function, the larger problem is that for all practical purposes the system is

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[17] When the first argument of an "Incomplete Beta function" as in equation (37) is 0 (i.e., as $m\to\infty$), then the function itself equals 0; when the first argument is 1 (i.e., when m=0), then the function is "complete" and equals $\Gamma(1/k+\delta/r)\Gamma(2-1/k)/\Gamma(2+\delta/r)$, where $\Gamma(\ldots)$ is the more familiar Gamma function. The ratio of the incomplete to the complete Beta function is the Beta cumulative distribution function from statistics, *when the Beta function's parameters — i.e.,* $1/k+\delta/r$ *and* $2-1/k$ *— are both positive.* This would permit plotting the form in *Excel*, by subverting that package's GAMMALN and BETA.DIST calls. Yet $0 < k < ½$ is the likely range of relative floor prices, so $2-1/k<0$. Fortunately, $\sum_{j=0}^{n}\binom{n}{j}\frac{\prod_{h=1}^{j}\left(\frac{1}{k}+\frac{\delta}{r}-h\right)\prod_{h=0}^{n-j-1}\left(2+\frac{\delta}{r}+h\right)}{\prod_{h=0}^{n-1}\left(2-\frac{1}{k}+h\right)}\frac{(-e^{-rm})^{j}}{(1-e^{-rm})^{n}}$ Beta[e^{-rm}, $1/k+\delta/r-j$, $2+n-1/k$] is a (clumsy) workaround for Beta[e^{-rm}, $1/k+\delta/r$, $2-1/k$], where n is the smallest integer for which $2+n-1/k>0$.

[18] Cf. Mood, Graybill, and Boes, *Introduction to the Theory of Statistics* (1974), pp. 198-202. Still, in its own terms, expression (42) solves the linear differential equation: $f'(\mathrm{L}) - \frac{r}{k}\left(\frac{1-k}{1-e^{-rL}}+k\right)f(\mathrm{L}) = -\delta\frac{r+\delta}{k}(1-k)^{2}e^{2rL}\big(k+(1-k)e^{rL}\big)^{-2-\delta/r}$.

not well identified. A statistician innocent of floor prices might fit a system such as (7)/(8) and (9b) — that is, with k=0 — in an internally consistent manner and find little amiss, apart from estimates of r that seem too high. The following table presents biased estimates of δ and r obtained by minimizing the Kullback-Leibler discrepancy between density (7) and the true ($k > 0$) but overlooked target density (37) for various true values of δ (i.e., .05 to .35 in increments of .05), r (i.e., .01 to .15 in increments of .02) and k (i.e., .01, .02, .05, .10, and .15):

Biased Estimates of δ and *r* when *k* is Neglected

k = .01

δ↓ *r*→	0.01	0.03	0.05	0.07	0.09	0.11	0.13	0.15
0.05	0.0495 0.0122	0.0498 0.0320	0.0499 0.0520	0.0499 0.0722	0.0499 0.0924	0.0499 0.1126	0.0499 0.1328	0.0500 0.1531
0.10	0.0989 0.0146	0.0993 0.0342	0.0995 0.0540	0.0996 0.0740	0.0997 0.0940	0.0998 0.1142	0.0998 0.1343	0.0998 0.1545
0.15	0.1483 0.0169	0.1486 0.0366	0.1490 0.0562	0.1492 0.0760	0.1493 0.0959	0.1495 0.1160	0.1495 0.1360	0.1496 0.1561
0.20	0.1977 0.0190	0.1980 0.0390	0.1984 0.0585	0.1987 0.0782	0.1989 0.0980	0.1991 0.1179	0.1992 0.1379	0.1993 0.1579
0.25	0.2472 0.0212	0.2473 0.0414	0.2477 0.0609	0.2481 0.0805	0.2484 0.1002	0.2486 0.1201	0.2488 0.1399	0.2489 0.1599
0.30	0.2967 0.0232	0.2967 0.0437	0.2971 0.0633	0.2975 0.0829	0.2978 0.1025	0.2981 0.1223	0.2983 0.1421	0.2985 0.1620
0.35	0.3462 0.0252	0.3460 0.0460	0.3464 0.0657	0.3468 0.0853	0.3472 0.1049	0.3475 0.1245	0.3478 0.1443	0.3480 0.1641

k = .02

δ↓ *r*→	0.01	0.03	0.05	0.07	0.09	0.11	0.13	0.15
0.05	0.0492 0.0142	0.0496 0.0340	0.0497 0.0541	0.0498 0.0745	0.0499 0.0949	0.0499 0.1153	0.0499 0.1357	0.0499 0.1562
0.10	0.0981 0.0186	0.0986 0.0382	0.0990 0.0580	0.0993 0.0780	0.0994 0.0981	0.0995 0.1184	0.0996 0.1387	0.0996 0.1591
0.15	0.1471 0.0228	0.1475 0.0427	0.1481 0.0623	0.1485 0.0820	0.1487 0.1019	0.1489 0.1220	0.1491 0.1422	0.1492 0.1624
0.20	0.1962 0.0267	0.1964 0.0472	0.1970 0.0667	0.1975 0.0863	0.1979 0.1060	0.1982 0.1259	0.1984 0.1459	0.1986 0.1660
0.25	0.2453 0.0306	0.2453 0.0516	0.2459 0.0712	0.2465 0.0907	0.2469 0.1103	0.2473 0.1300	0.2476 0.1499	0.2479 0.1699
0.30	0.2945 0.0344	0.2943 0.0559	0.2948 0.0757	0.2954 0.0952	0.2959 0.1147	0.2964 0.1343	0.2967 0.1541	0.2971 0.1739
0.35	0.3437 0.0381	0.3437 0.0591	0.3437 0.0801	0.3443 0.0997	0.3448 0.1192	0.3453 0.1387	0.3458 0.1584	0.3462 0.1781

k = .05

δ↓ *r*→	0.01	0.03	0.05	0.07	0.09	0.11	0.13	0.15
0.05	0.0484 0.0201	0.0490 0.0401	0.0494 0.0607	0.0495 0.0816	0.0496 0.1026	0.0497 0.1238	0.0498 0.1450	0.0498 0.1662
0.10	0.0964 0.0299	0.0971 0.0502	0.0978 0.0701	0.0983 0.0904	0.0986 0.1110	0.0988 0.1318	0.0990 0.1527	0.0991 0.1737
0.15	0.1446 0.0393	0.1451 0.0603	0.1459 0.0802	0.1466 0.1001	0.1471 0.1203	0.1475 0.1407	0.1479 0.1613	0.1481 0.1820
0.20	0.1929 0.0484	0.1931 0.0703	0.1939 0.0904	0.1947 0.1102	0.1954 0.1302	0.1959 0.1503	0.1964 0.1706	0.1968 0.1911
0.25	0.2413 0.0575	0.2411 0.0801	0.2418 0.1005	0.2426 0.1204	0.2434 0.1403	0.2441 0.1602	0.2447 0.1803	0.2452 0.2005
0.30	0.2897 0.0664	0.2892 0.0897	0.2898 0.1106	0.2906 0.1306	0.2914 0.1505	0.2922 0.1703	0.2929 0.1903	0.2935 0.2103
0.35	0.3381 0.0753	0.3374 0.0992	0.3378 0.1205	0.3385 0.1408	0.3394 0.1607	0.3402 0.1805	0.3409 0.2004	0.3416 0.2203

Biased Estimates of δ and *r* when *k* is Neglected
(continued)

k = .10 δ↓ *r*→	0.01	0.03	0.05	0.07	0.09	0.11	0.13	0.15
0.05	0.0475 0.0300	0.0483 0.0511	0.0488 0.0727	0.0491 0.0948	0.0493 0.1171	0.0494 0.1396	0.0495 0.1622	0.0496 0.1848
0.10	0.0946 0.0489	0.0954 0.0706	0.0963 0.0916	0.0970 0.1129	0.0975 0.1345	0.0979 0.1563	0.0982 0.1784	0.0984 0.2006
0.15	0.1418 0.0673	0.1424 0.0900	0.1433 0.1112	0.1442 0.1321	0.1450 0.1533	0.1456 0.1747	0.1461 0.1963	0.1465 0.2181
0.20	0.1892 0.0855	0.1894 0.1091	0.1903 0.1307	0.1913 0.1517	0.1922 0.1727	0.1930 0.1938	0.1937 0.2150	0.1943 0.2365
0.25	0.2366 0.1037	0.2366 0.1279	0.2373 0.1500	0.2382 0.1713	0.2392 0.1923	0.2401 0.2132	0.2410 0.2343	0.2417 0.2555
0.30	0.2839 0.1218	0.2837 0.1466	0.2843 0.1692	0.2852 0.1907	0.2862 0.2118	0.2872 0.2328	0.2881 0.2538	0.2889 0.2748
0.35	0.3314 0.1398	0.3310 0.1651	0.3314 0.1881	0.3322 0.2100	0.3332 0.2313	0.3341 0.2524	0.3351 0.2733	0.3360 0.2943

k = .15 δ↓ *r*→	0.01	0.03	0.05	0.07	0.09	0.11	0.13	0.15
0.05	0.0468 0.0409	0.0478 0.0635	0.0484 0.0864	0.0488 0.1100	0.0490 0.1338	0.0492 0.1579	0.0493 0.1821	0.0494 0.2064
0.10	0.0933 0.0697	0.0942 0.0932	0.0951 0.1156	0.0959 0.1383	0.0966 0.1613	0.0970 0.1846	0.0974 0.2081	0.0977 0.2318
0.15	0.1399 0.0982	0.1405 0.1226	0.1415 0.1454	0.1425 0.1678	0.1434 0.1904	0.1441 0.2132	0.1447 0.2361	0.1452 0.2593
0.20	0.1865 0.1265	0.1869 0.1517	0.1878 0.1750	0.1888 0.1976	0.1898 0.2200	0.1907 0.2426	0.1915 0.2652	0.1922 0.2880
0.25	0.2331 0.1550	0.2333 0.1805	0.2341 0.2043	0.2351 0.2272	0.2362 0.2498	0.2372 0.2723	0.2381 0.2947	0.2389 0.3173
0.30	0.2797 0.1832	0.2799 0.2091	0.2805 0.2334	0.2814 0.2567	0.2825 0.2795	0.2835 0.3020	0.2845 0.3245	0.2854 0.3469
0.35	0.3264 0.2113	0.3264 0.2377	0.3270 0.2624	0.3278 0.2860	0.3288 0.3090	0.3298 0.3317	0.3308 0.3542	0.3318 0.3767

Except for a few nonmonotonic estimates of high-end δ as *r* passes from .01 to .05 (shaded yellow), the table's entries are well behaved, and their message is clear: neglecting *k* does little harm to estimates of δ but biases estimates of *r* upward, particularly for large δ and large *k*. Without convincing data on *k*, an age-price grid-search corrected by the numerical survival function implied by density (37) would need to have *r* pre-set to some reasonable value, in hopes of fitting δ and *k*.

Now, a too-high value of *r*, taken literally, implies discounting too close to the no-tomorrow limit. Yet the imposition of a floor-price *does* cut the farthest tomorrows off every individual asset, for which the best response would have the flavor of extra discounting.[19] The common-sense view that premature retirements must imply faster depreciation should be reconsidered in light of this ersatz discounting. This leads to a question: For a given value of *r* and an effectively fixed distribution of zero-price *lifespans*, what happens to the depreciation rate if a positive floor-price is imposed where before there had been none? To approximate an answer, choose a value of δ to minimize the Kullback-Leibler discrepancy between the lifespan density (38) and *target* density (7), where the target's δ is already known and fixed. Per the scant bias in δ just found, there is little reason to suspect much change in the

[19] For example, the best fit of the individual blue schedule in plot (32), all the way down to 0 at age 10, against the innocent age-price form $\frac{e^{\tilde{r}s}-e^{\tilde{r}10}}{1-e^{\tilde{r}10}}$, finds a *faux* $\tilde{r}$ of about .168, well above the given $r = .07$. *Faux* $\tilde{r}$ vary across individual draws of *m*, with shorter *m* leading to higher $\tilde{r}$.

cohort depreciation rate, and indeed that is the case. The following table shows that values of δ needed to hold the lifespan distribution roughly invariant when a positive floor-price is introduced are little different from their zero floor-price counterparts. (For $k < .05$, differences are indiscernible.)

Without Obsolescence, δ Changes Little when *k* is Introduced

k = .05								
δ↓ *r*→	0.01	0.03	0.05	0.07	0.09	0.11	0.13	0.15
0.05	0.0501	0.0501	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
0.10	0.1002	0.1002	0.1001	0.1001	0.1001	0.1001	0.1001	0.1001
0.15	0.1503	0.1503	0.1502	0.1502	0.1502	0.1502	0.1501	0.1501
0.20	0.2004	0.2004	0.2003	0.2003	0.2003	0.2002	0.2002	0.2002
0.25	0.2506	0.2505	0.2504	0.2504	0.2504	0.2503	0.2503	0.2503
0.30	0.3007	0.3006	0.3005	0.3005	0.3005	0.3004	0.3004	0.3004
0.35	0.3508	0.3507	0.3507	0.3506	0.3506	0.3505	0.3505	0.3505

k = .10								
δ↓ *r*→	0.01	0.03	0.05	0.07	0.09	0.11	0.13	0.15
0.05	0.0504	0.0502	0.0502	0.0501	0.0501	0.0501	0.0501	0.0501
0.10	0.1008	0.1006	0.1005	0.1004	0.1004	0.1003	0.1003	0.1003
0.15	0.1513	0.1511	0.1509	0.1508	0.1507	0.1506	0.1506	0.1505
0.20	0.2018	0.2015	0.2013	0.2012	0.2011	0.2010	0.2009	0.2008
0.25	0.2523	0.2520	0.2518	0.2516	0.2515	0.2514	0.2513	0.2512
0.30	0.3028	0.3025	0.3022	0.3020	0.3019	0.3017	0.3016	0.3015
0.35	0.3533	0.3530	0.3527	0.3525	0.3523	0.3522	0.3520	0.3519

k = .15								
δ↓ *r*→	0.01	0.03	0.05	0.07	0.09	0.11	0.13	0.15
0.05	0.0508	0.0505	0.0504	0.0503	0.0503	0.0502	0.0502	0.0502
0.10	0.1019	0.1014	0.1012	0.1010	0.1009	0.1008	0.1007	0.1006
0.15	0.1530	0.1525	0.1521	0.1518	0.1516	0.1515	0.1513	0.1512
0.20	0.2042	0.2035	0.2031	0.2027	0.2025	0.2023	0.2021	0.2019
0.25	0.2550	0.2546	0.2541	0.2537	0.2534	0.2531	0.2529	0.2527
0.30	0.3050	0.3057	0.3051	0.3047	0.3043	0.3040	0.3038	0.3036
0.35	0.3559	0.3568	0.3562	0.3557	0.3553	0.3550	0.3547	0.3544

One can't fault designers of capital-retirement surveys for not including questions about scrap values in an era when obsolescence didn't matter much.

A drawback of allowing a positive floor-price is that it induces a mismatch between a thoroughly geometric cohort resale-price profile, as per (35) or (39), and a *rental*-price profile that declines faster than geometric. This is a consequence of any individual's "retirement bond," $q_0\, k\, e^{-r(m-s)}$ — i.e., from the line just below expression (34) — having a rental value of zero:

$$r\, q_0\, k\, e^{-r(m-s)} - \partial(q_0\, k\, e^{-r(m-s)})/\partial s = 0.$$

Weight individual-level rental prices in their productive years (33) by (37) for the cohort rental-price:

$$q_0\,(r+\delta)\,\Phi(s) = \int_s^\infty q_0 r \frac{e^{rm}-k}{e^{rm}-1}\, e^{rm}\,(e^{rm}-1)^{1/k-1}\,\mathrm{Beta}[e^{-rm},\, 1/k+\delta/r,\, 2-1/k]\;\delta(\delta+r)/(rk)\; dm \qquad (40)$$

This indeed starts at $q_0\ (r+\delta)\ e^{-\delta s}$ at age s=0, confirming the standard geometric user-cost. Then it falls away, converging ultimately on $q_0\ (r+\delta)\frac{r}{r+k\delta}\ e^{-\delta s}$ at asymptotic rate r. Fortunately, a weighted sum of several exponential terms approximates the rental-price profile reasonably well:

$$q_0\ (r+\delta)\ \Phi(s) \approx q_0 \frac{r+\delta}{r+k\,\delta}\Big(r + k\ \delta\ (w_1\, e^{-\beta_1 s} + w_2\, e^{-\beta_2 s} + (1 - w_1 - w_2)e^{-rs}\Big)\ e^{-\delta s} \tag{41}$$

...where short, intermediate, and asymptotic rates of convergence (i.e., $\beta_1 > \beta_2 > r$) rot in turn, leaving the converged-on remnant profile. For depreciation-rate, discount-rate and floor-price grid-values:

$$\delta \times r \times k =$$
$$\{.01, .07, .13, .19, .25, .31, .37\} \times \{.01, .05, .09, .13, .17, .21, .25\} \times \{.025, .05, .075, .1, .125, .15, .175, .2\}$$

...a pooled nonlinear regression of each numerical evaluation of (40) across 101 equally-spaced ages from 0 to $ln(1000)/\delta$[20] for each (δ,r,k)-combination generates weights and rates:

$$w_1 = -.186 \qquad\qquad w_2 = .560$$
$$\beta_1 = 3r + .0714\ \delta^{.516} r^{1.43} k^{-1.34} \qquad\qquad \beta_2 = 2r + 9.43\ \delta^{.902}\ r^{.133} k^{1.43}$$

...that are sufficient to make (41) a good approximation to (40). For the approximate cohort age-efficiency profile, divide (41) by $q_0(r+\delta)$.[21]

6. Individual and Cohort Cases, with a Positive Scrap Price, Inflation, and Obsolescence (...Difficult)

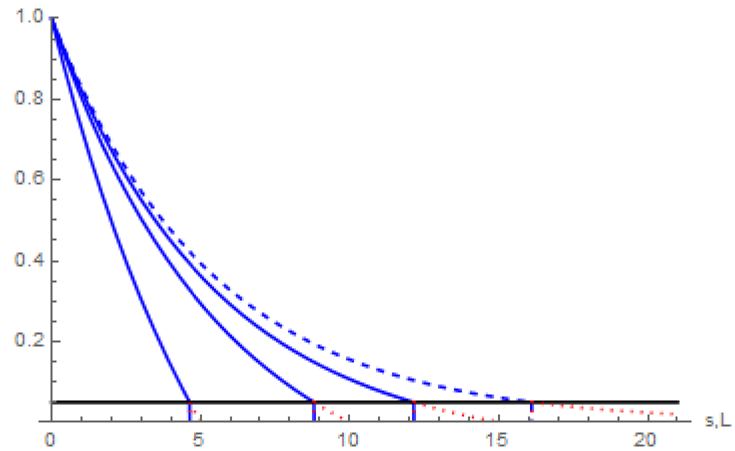
Now the math can just about describe the common-sense idea that obsolescence induces early retirements. It can also assess the effects of early, obsolescence-driven retirements on depreciation and deterioration. Adjusting (15a/15b/15c) from section 3 to admit a floor-price — alternatively, adjusting (31) from section 5 to allow inflation and quality improvement — will bring us to right-truncated nearly geometric cohort-level resale-price profiles and faster-than-geometric cohort-level rental-price profiles. But tractable analytic results will not often obtain here, so we will turn to numeric demonstrations. To begin, reconsider a plot like (32), with r explicitly parted into the difference between the nominal rate of return (i) and the expected naive inflation rate (π); *also* admit a constant positive anticipated-and-realized rate of obsolescence (b). Assigning realized nominal inflation to the new-supply price amounts to the treatment shown in the middle set of plots in (18), on which we'll focus to take advantage of a floor-price schedule that stays flat.[22] The plot just below redraws age-price profiles for four individuals. All are retired when their resale values have declined to k = 5 percent of the frontier supply price:

---

[20] At age $ln(1000)/\delta$, the cohort's resale value has depreciated to a thousandth of its starting value.

[21] The technique of approximating a non-geometric profile by a linear combination of geometric pieces has potentially wide application, making non-geometric cohorts about as easy to compute as geometric ones. Depreciation researchers wishing a least-action statistical agency to implement their findings would be well advised to supply the multi-geometric approximation, with well-chosen rates and weights.

[22] That is, the relevant new-investment price in this section is $q_0(t)$, equivalently $q_0(v)e^{\pi(t-v)}$.



(42)

Without a floor value (i.e., for *k*=0), the first three individuals would have been held to their full lifespans of 5, 10, and 15 years, respectively. But obsolescence transforms what had been downwardly concave resale-price profiles into downwardly convex forms, particularly for long-lived individuals, leading to long gaps between retirement ages and physical lifespans. In fact, the fourth individual in the plot, its path given by the dashed blue line, is retired just after age 16 but has no finite lifespan. To describe an obsolescing individual's resale-price profile vis-à-vis the persistently inflating frontier new-supply price, $q_0(t) = q_0(v)e^{\pi(t-v)}$, write:

$$\begin{cases} e^{-b(t-v)} \dfrac{e^{(i-\pi+b)s} - \frac{e^{(i-\pi)m}-k}{e^{-bm}-k}}{1-\frac{e^{(i-\pi)m}-k}{e^{-bm}-k}} & for\ s\ between\ 0\ and\ m \\ 0 & for\ s \geq m \end{cases} \qquad (43a)$$

equivalently:

$$\begin{cases} e^{-b(t-v)} \dfrac{e^{(i-\pi+b)s} - e^{(i-\pi+b)L}}{1-e^{(i-\pi+b)L}} & for\ s\ between\ 0\ and\ such\ m\ that\ ln\left[\frac{e^{(i-\pi)m}-k}{e^{-bm}-k}\right]/(i-\pi+b) = L \\ 0 & for\ s \geq m \end{cases} \qquad (43b)$$

For *b*=0, form (43a) becomes (31), and the description of retirement age *m* in (43b) agrees with note 16. However, for *b*>0, the equivalence break down: the boundary condition on *m* needs $e^{-bm} > k$ for *L* to stay Real,[23] so the dashed blue line in (42) attains the floor price at the largest allowable value of *m*: $-ln(k)/b$. Individual rental profiles that correspond to (43a) are:

$$\begin{cases} (i-\pi+b)e^{-b(t-v)} \dfrac{e^{(i-\pi)m}-k}{e^{(i-\pi)m}-e^{-bm}} & for\ s\ between\ 0\ and\ m \\ 0 & for\ s \geq m \end{cases} \qquad (44a)$$

equivalently, for (43b):

$$\begin{cases} e^{-b(t-v)} \dfrac{i-\pi+b}{1-e^{-(i-\pi+b)L}} & for\ s\ between\ 0\ and\ such\ m\ that\ ln\left[\frac{e^{(i-\pi)m}-k}{e^{-bm}-k}\right]/(i-\pi+b) = L \\ 0 & for\ s \geq m \end{cases} \qquad (44b)$$

---

[23] I'm assuming $i > \pi$ and $1 > k \geq 0$ throughout, so $e^{(i-\pi)m} - k$ in the numerator is safely positive.

Again the floor-price induces a retirement bond; the present value of an individual rental-price profile:

$$\int_s^m e^{-(i-\pi)(u-s)}\ (i-\pi+b)e^{-b\,u}\frac{e^{(i-\pi)m}-k}{e^{(i-\pi)m}-e^{-bm}}du$$

...falls short of the resale-price profile (43a) by $k\ e^{-(m-s)(i-\pi)}$, which earns no rent.

We have already worked out how $k > 0$ gives rise to a faster-than-geometric cohort rental-price dual to the geometric resale-price profile, before obsolescence. Now for $b > 0$ on top of $k > 0$, problems are compounded, for the finite maximal value for m nudges even the resale-price profile away from an exactly geometric form. In terms of retirement ages, a geometric age-price cohort would imply a density $f(m)$ that is in principle observable and would solve:

$$e^{-(\delta+b)s} = \int_s^{-ln(k)/b} e^{-bs}\frac{e^{(i-\pi+b)s}-\frac{e^{(i-\pi)m}-k}{e^{-b\,m}-k}}{1-\frac{e^{(i-\pi)m}-k}{e^{-b\,m}-k}}\ f(m)\ dm \tag{45}$$

...where the upper limit of integration imposes the restriction that keeps L Real. Equivalently, the unobservable density of lifespans $f(L)$ would solve:

$$e^{-(\delta+b)s} = \int_{\ln\left(\frac{e^{(i-\pi)s}-k}{e^{-bs}-k}\right)/(i-\pi+b)}^{\infty} e^{-bs}\frac{e^{(i-\pi+b)s}-e^{(i-\pi+b)L}}{1-e^{(i-\pi+b)L}}\ f(L)\ dL \tag{46}$$

...where the lower limit of integration spells out the value of L that obtains for the next individual to reach retirement. For that retirement age (i.e., s=m), the Jacobian is $J = \partial\left[ln\left(\frac{e^{(i-\pi)m}-k}{e^{-bm}-k}\right)/(i-\pi+b)\right]/\partial m = \left((i-\pi)/(1-ke^{-(i-\pi)m}) + b/(1-ke^{bm})\right)/(i-\pi+b)$, so $f(m) = J\ f(L)$. An inverse Jacobian, explicitly in terms of L, is not available, though $1/J$ will serve numerically.

Applying the variational exercise twice to (45), then tidying up, gives the linear differential equation:

$$f'(m) + \left(\frac{i-\pi+b}{k}\frac{1}{e^{-(i-\pi)m}-e^{bm}}+\frac{b}{1-e^{-(i-\pi+b)m}}+\frac{i-\pi}{e^{(i-\pi+b)m}-1}\right)f(m) = -e^{-(\delta+b)m}(\delta+i-\pi+b)\delta/k. \tag{47}$$

This solves as:

$$f(m) = \frac{\frac{\delta}{k}(\delta+i-\pi+b)\int_m^T Exp\left[Beta\left[e^{-(i-\pi+b)x},\ \frac{b}{i-\pi+b},\ 0\right]/k\right]\left(e^{bx}-e^{-(i-\pi)x}\right)e^{-(\delta+b)x}dx+\mathrm{C}}{Exp\left[Beta\left[e^{-(i-\pi+b)m},\ \frac{b}{i-\pi+b},\ 0\right]/k\right]\left(e^{bm}-e^{-(i-\pi)m}\right)} \tag{48}$$

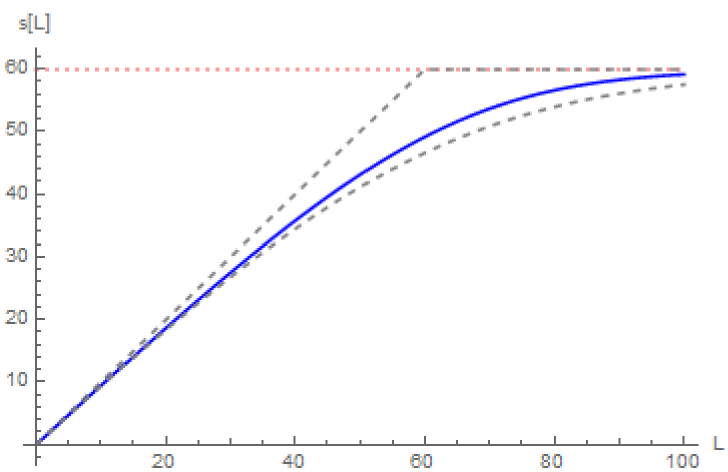
...where the upper-limit of integration T and the constant C must be set. Choosing $T \to \infty$ and C=0 gives the tantalizing and proper solution to the wrong problem — i.e., (45), with its upper limit of integration relaxed from $-ln(k)/b$ to ∞. Not only does the relaxation make $f(L)$ *im*proper, it implies continuances of any individual age-price profiles as might have $m > -ln(k)/b$ — that is, red-dotted right tails of any blue lines that might be drawn to the *right* of the dashed line in (42) — that do not reach zero, but increase.

Choosing $T = -ln(k)/b$ preserves $f(L)$'s propriety but causes unavoidable problems of its own, for now a given C will satisfy (45) exactly at just one age, rather than over the whole interval $0 \le s \le -ln(k)/b$. A compromise takes the C that minimizes the sum of squared differences between the left- and right-sides of (45) over a fine, evenly-spaced grid of ages from 0 to $-ln(k)/b$. (The compromise value depends on the system parameters δ, $i-\pi$, b, and k.)[24]

In the L-domain, the linear differential equation derived from (46) is less attractive:

$$f'(L) + (i-\pi+b)\left(\frac{b\, e^{(i-\pi+b)L} - (i-\pi)e^{(i-\pi+b)s}}{b\, e^{(i-\pi+b)L} + (i-\pi)e^{(i-\pi+b)s}} - \frac{e^{(i-\pi)s} - k}{e^{(i-\pi+b)L} - 1}\Big/k + b(i-\pi)e^{bs}\frac{\left(e^{(i-\pi)s} - k\right)\left(e^{(i-\pi+b)s} - e^{(i-\pi+b)L}\right)}{\left(b\, e^{(i-\pi+b)L} + (i-\pi)e^{(i-\pi+b)s}\right)^2}\right) f(L)$$
$$= -e^{(b-\delta)s}\left(\frac{e^{(i-\pi)s} - k}{b\, e^{(i-\pi+b)L} + (i-\pi)e^{(i-\pi+b)s}}\right)^2 (\delta+i-\pi+b)\,(i-\pi+b)^2\,\delta/k \qquad (49)$$

...because s is an implicit inverse function of L (though well behaved). In the example plot just below, given, say, parameter values **δ =.10**, ***i* – π =.03**, ***b* =.05**, and ***k* =.05**, the solid-blue implicit function $s(L)$ rises between dashed-gray limiting cases[25] on its way to the dotted asymptote at $-ln(k)/b$:



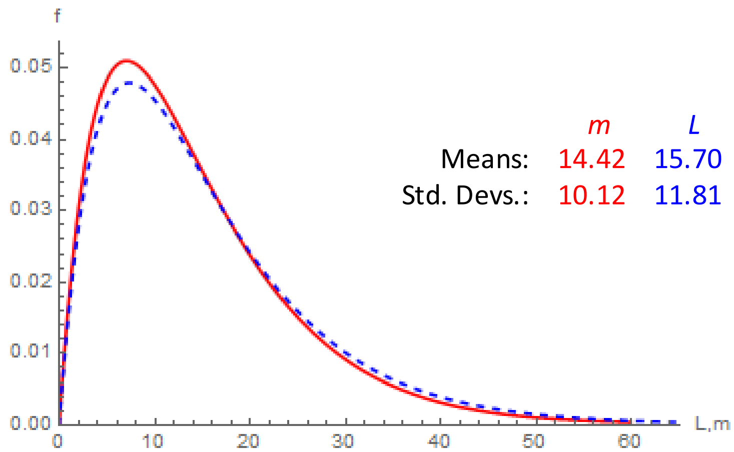
Solving (49) numerically requires choosing a convenient initial condition rather than a constant of integration, but the same need for an approximating compromise crops up, and it is resolved as before: pick the initial condition to minimize the sum of squared differences between the left- and right-hand sides of (46) across a fine, evenly-spaced grid of ages from 0 to (about) $-ln(k)/b$. By renumbering and applying a Jacobian transformation — i.e., replace $(m, f(m))$ pairs that approximately solve (45) by $\mathbf{(}ln\left(\frac{e^{(i-\pi)m} - k}{e^{-bm} - k}\right)/(i-\pi+b),\ f(m)/\mathrm{J}\mathbf{)}$, or replace $(L, f(L))$ pairs that nearly solve (46) by $(s(L), \mathrm{J}\, f(L))$ — one can verify that the approximate solutions to (45) and (46) agree to tight tolerances.

---

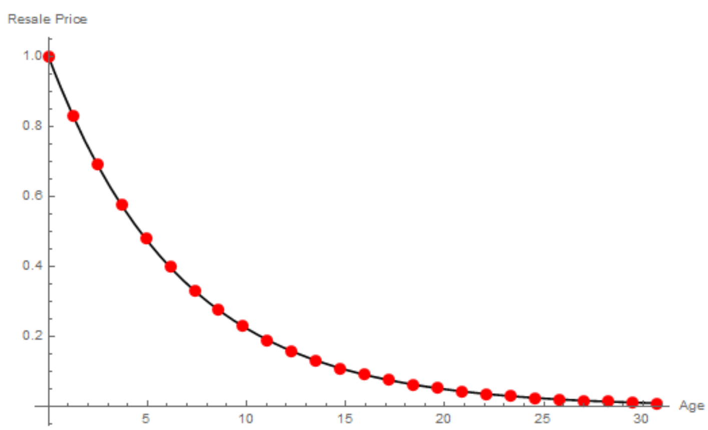
[24] The compromise leaves $f(m)$ somewhat improper, integrating to slightly more than 1. *Cf.* footnote 26, below.

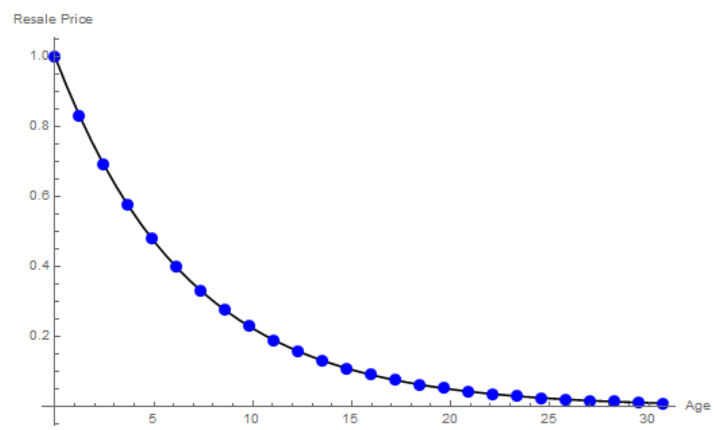
[25] ...which *are* closed form: for $i-\pi \to \infty$, $s(L) = L$ for $L \le -ln(k)/b$ but $-ln(k)/b$ otherwise; while for $i-\pi = 0$, $s(L) = L - ln[1 + k(e^{bL} - 1)]/b$. For large L, $s(L)$ closes the gap with $-ln(k)/b$ at rate $-(i-\pi+b)$.

For these moderate exemplary parameter values, there is not much difference between the densities of retirement ages $f(m)$ (solid red in the following plot) and lifespans $f(L)$ (dashed blue):



The right-truncation of $f(m)$ is barely discernible at $-ln(k)/b \approx 60$ years, while the wisps of $f(L)$ continue, in principle, indefinitely. Both densities meet their geometric resale-price profiles to visual tolerances:[26]





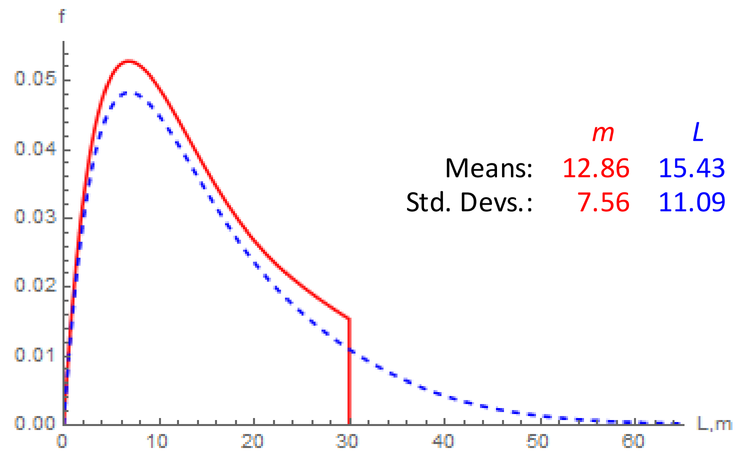
Next, suppose the rate of quality improvement / obsolescence doubles, to ***b* =.10**, permanently. The maximal service-life is halved, to $-ln(k)/b \approx 30$ years: the shape of $f(m)$ should change accordingly, but the shape of $f(L)$ should stay about the same. To implement these considerations, alter δ to nearly satisfy the *revised* fine, evenly-spaced grid of ages from 0 to almost $-ln(k)/b$, such that the Kullback-Leibler discrepancy between the revised density $f(L)$ and the original solution to (46) is minimized.[27] The

[26] For $f(m)$, the integral conditions work out to 1.0000277, not 1 exactly, at age 0; and to .009957, not .01, at age $ln(100)/(\delta+b) \approx 30.701$. For $f(L)$, the corresponding conditions are 1.00000456 and .009946, respectively. For both densities, departures from the targeted $e^{-(\delta+b)s}$ run monotonically from "too high" to "too low" as s increases, so the red or blue dots in (59) are ever-so-slightly too steep. By age $-ln(k)/b \approx 60$ years, the final retirement occurs, so a hypothetical dot would hit zero; the geometric approximation would have failed before then.

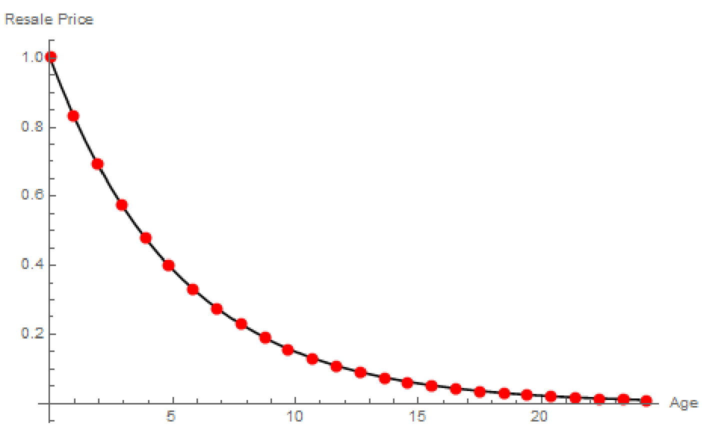
[27] In practice, for the new, doubled value of b, I solved (47)/(48) for $f(m)$ repeatedly, once each for a sequence of δ test-values, carried out the Jacobian transformation to $f(L)$ each time, and computed the K-L discrepancy against the original $f(L)$ for each candidate-$f(L)$, until reaching a minimum.

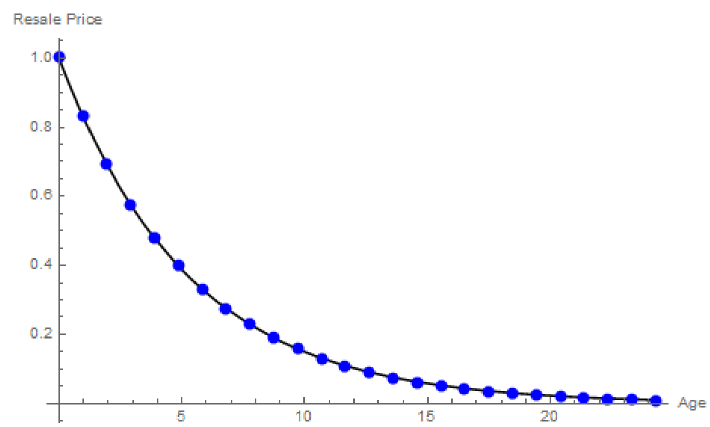
resulting best-choice δ = .0896441, implies a clawback of a fifth the change in *b*. Further, the same experiment, against a backdrop of a *zero* floor-price, arrived at a best-choice δ = .0908456.[28] So a floor-price marginally worsens the clawback, as premature retirements activate the channel of *ersatz* extra-discounting.[29]

For completeness, the best-choice *f*(*m*) and *f*(*L*), now that *b*=.10 (and holding *k*=.05), plot as:



...so premature retirements are indeed substantial, while the distribution of *lifespans* is little changed.[30] Again, the integral conditions are met almost exactly:





What about cohort-level rental prices? Absent obsolescence (*cf*. pp. 17-8), the presence of a floor price made the cohort-level rental profile fall faster initially than its associated geometric cohort-

[28] This is easier to carry out, as the problem has a closed form when *k*=0:

$$Min_{\delta} \int_0^{\infty} e^{-\delta L}\left(1-e^{-(.03+.1)\delta L}\right)\delta^{\frac{\delta+.03+.1}{.03+.1}} ln\left[\frac{e^{-\delta L}\left(1-e^{-(.03+.1)\delta L}\right)\delta^{\frac{\delta+.03+.1}{.03+.1}}}{e^{-.1L}\left(1-e^{-(.03+.05).1L}\right).1^{\frac{.1+.03+.05}{.03+.05}}}\right]dL$$

[29] *Cf*. again the small table on p. 17, where increases in *k*, for *b*=0, marginally *increase* δ. This suggests a "sweet spot," where a change in obsolescence would pass through to the overall depreciation rate unimpeded. I have neither looked for nor found it, though.

[30] To retire all a cohort's members by age 8 — so the *average* retirement age approximates the *simultaneous* quit age of 3 or 4 suggested by Diewert and Wei — for *b*=.18611 (*cf*. p. 8 above) requires *k* =.226. That feels high.

level resale-price profile. *With* obsolescence, we have found the approximate density of retirement ages is truncated. This leads to a cohort-level rental profile that is, if anything, slightly too large at age s=0,[31] then conventionally well behaved over a satisfactorily broad middle range of the cohort's years in service, and fully dead at age $-ln(k)/b$. Weight individual-level rental prices in their productive years (44a) by (48) to find the cohort rental-price:

$$q_0(\mathrm{t})\,(i-\pi+\delta+b+\mathbf{x})\,\Phi(\mathrm{s}) = \tag{50}$$
$$q_0(t)\int_s^{-ln(k)/b}(i-\pi+b)e^{-bs}\frac{e^{(i-\pi)m}-k}{e^{(i-\pi)m}-e^{-bm}}\left\{\frac{\frac{\delta}{k}(\delta+i-\pi+b)\int_m^{-\ln(k)/b}Exp\left[Beta\left[e^{-(i-\pi+b)x},\frac{b}{i-\pi+b},0\right]/k\right]\left(e^{bx}-e^{-(i-\pi)x}\right)e^{-(\delta+b)x}dx+\mathbf{C}}{Exp\left[Beta\left[e^{-(i-\pi+b)m},\ \frac{b}{i-\pi+b},\ 0\right]/k\right]\left(e^{bm}-e^{-(i-\pi)m}\right)}\right\}dm$$

This is decently approximated as a weighted sum of two exponentials:

$$\approx q_0(\mathrm{t})\,\frac{i-\pi+\delta+b+\mathbf{x}}{1-k^{1+(\delta-g)/b}}\,e^{-(\delta+b)s}\left(1-(k^{1/b}e^{s})^{\delta+b-g}\right) \tag{51}$$

...where $\mathbf{x} = .699\,\delta^{-.649}\,r^{-.155}\,b^{1.89}\,k^{1.54}$ and $g = -.157 + 1.69\,\delta - .292\,r - .189\,b + .533\,k$.[32] Unlike the rental-price profile in (40)/(41), which fell quickly from the geometric standard $(i-\pi+\delta)e^{-\delta s}$ toward a lower level, all while hewing toward the same asymptotic geometric rate of decline $-\delta$, profile (50)/(51) compares well with $(i-\pi+\delta+b)e^{-(\delta+b)s}$ through most of its career, only crashing toward the end. The difference is a substantial, positive b, which suggests there must be some small b (between 0 and .04) for which $(i-\pi+\delta+b)e^{-(\delta+b)s}$ is unobjectionable.[33] For computers, b is larger than that, but k is unknown.

The difficulties of fitting the probability density function of retirement ages (48) in the presence of obsolescence, inflation, and a floor price should be apparent; and the same risks of constructing Hulten-Wykoff retirement correctives without first imposing sure values on some of the parameters, would dog this effort even more than the previous attempt in the b=0 case (37) — and all for a basic shape that a right-truncated Gamma density could approximate pretty well.[34] I don't pursue it further.

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[31] That is, too large compared to $(r+\delta+b)e^{-(\delta+b)s}$, so I've included an **x** for "extra" on the left side of (50), to keep $\Phi(0) = 1$. The excess is due to the compromise choice of the constant of integration ***C*** for $f(m)$ in equations (48)/(50), which makes the integral $\int_0^{-ln(k)/b} f(m)dm$ come out slightly above 1.

[32] The ten coefficients that give structure to **x** and g are taken from the best fit of a single nonlinear regression of the right side of (50) against (51) over the parameter grid:

$$\begin{array}{ccccccccc} \delta & \times & i-\pi & \times & b & \times & k & \times & s \\ \{.01, .07, .13, .19, .25, .31, .37\} & \times & \{.01, .03, .05, .09, .13, .17\} & \times & \{.04, .08, .12, .16, .2\} & \times & \{.025, .05, .075, .1, .125, .15\} & \times & \{\boldsymbol{s}\}, \end{array}$$

...where vectors $\{\boldsymbol{s}\}$ are each 101 equally-spaced ages from 0 through $-ln(k)/b$, inclusive; so the regression had 127,260 observations. The functional form of (51) constrains the cohort rental price to $q_0(\mathrm{t})(i-\pi+\delta+b+\mathbf{x})$ at age 0 but to 0 at age $-ln(k)/b$. For the cohort age-efficiency profile, divide through by $q_0(\mathrm{t})(i-\pi+\delta+b+\mathbf{x})$.

[33] *Cf.* footnote 29, above.

[34] The only problem with the Gamma is unpacking all 4 effects — δ, $i-\pi$, b, and k — from just two parameters.

7. Where Obsolescence "Fits" and the Role of Disaggregate Information in Geometric Stocks

This essay may be read as a bottom-up response to Diewert and Wei's "Getting Rental Prices Right for Computers" (2015), which demonstrated an essential compatibility between standard geometric accounting of wealth and productive stocks of an asset type, and thoroughly *non*-geometric cohorts of the same type, provided a few rates hold roughly constant over a long-enough stretch of time. The cohorts in question were *simultaneously* one-hoss shay — every member of a cohort is scrapped at the same age — an extreme position that both simplifies the math *and* implies that more realistic (i.e., less non-geometric) cohort patterns would be subsumable into geometric stocks as well.[35]

But the cohorts in question were computers, where the primary mode of depreciation is not breaking (whether simultaneously across individuals or not) but being partly outclassed by the latest entry, and then *further* outclassed by the entry after *that*. This is one description of obsolescence, and though it is not an encompassing definition of that slippery term, it is the sort that comes quickest to mind. "Getting Rental Prices Right" cites obsolescence as the reason for quick extinguishment of service-flows, but beyond that does not wrestle with the phenomenon. (The potential of obsolescence to overstate the user-cost via jacking up the revaluation term is suggested in a footnote.)

By operationalizing obsolescence as the reduced price an old individual would fetch once restored to its original, "as-if-new" condition in a market with other individuals that really *are* new, and invoking a Law of One Price across all putatively restored members of an old cohort, we can distinguish obsolescence (what happens across cohorts) from "net" depreciation (what happens across individuals within a cohort) even if business owners cannot. The former is plausibly modeled as a geometric process; the latter probably not, at the individual level. The former may be inferred by hedonic methods; the latter, for an unavoidable, essentially fixed distribution of service-lives, by scrapyard statisticians, in simple cases. Both sources of information would usefully supplement standard age-price regressions.

Characterizing a distribution of service-lives as fixed and unavoidable is sure to raise hackles; what is really fixed is the joint distribution of management styles, maintenance practices, hard use, and sloth. Yet age-price studies are conducted only occasionally, and their findings are usually taken as fixed or very durable, whether by agencies that cite such studies' cohort-level results to justify their geometric stock-level accounting, or by agencies that compare them to their own vintage accounts built up from never-changing individual-level forms and assumed distributions. Whether at stock, cohort, or individual levels, the working assumption of fixed distributions is all around us.

This essay's innovation has been to use the assumption of approximately-fixed service-life distributions to identify the clawback in net depreciation δ when newly (and permanently) speeded obsolescence *b* raises the effective discount rate. At the possibly geometric cohort level and (thanks to Diewert and Wei) at the likely geometric stock level, the components of $\delta+b$ are not easily separated. Yet an individual-to-cohort examination has shown that δ partly offsets *b*. The specifics of that examination — one-hoss shay age-efficiency forms among individuals within a cohort, geometric

[35] The introduction of "Constant Efficiency Profile" cohorts, later in their paper, drives home the point.

obsolescence across cohorts, and combined overall geometric depreciation for any cohort — made inferring the service-life distributions, and holding them almost fixed, tractable. Yet partial clawback should characterize any aggregation of fixed individual resale-price profiles to a fixed cohort resale-price profile, provided individual profiles respond to discounting. (Straightline individual forms do not.)

Admitting a positive floor-price to the analysis adds realism, for now faster obsolescence does hasten retirements (which happen sooner than physical lifespans). It also greatly complicates the math, and it cleaves cohort age-price price profiles that are geometric or nearly so from cohort age-efficiency profiles that are not. (Weighted sums of two or more geometric profiles can reliably represent the latter.) But the basic story still holds: net depreciation partly offsets changes in obsolescence, through any mechanism that reduces tomorrows, whether ordinary discounting or early retirements.