Counting Multidimensional Deprivations in the Presence of Differences in Needs

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Counting multidimensional deprivations in the presence of differences in needs

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Abstract

Individuals from different demographic population subgroups and households of different size and composition exhibit different needs. Multidimensional deprivation comparisons in the presence of these differences in needs have yet to be analysed. This paper proposes a family of multidimensional deprivation indices that explicitly takes into account observed differences in needs across demographically heterogeneous units (i.e., either households of different size and composition or individuals of different population subgroups). The proposed counting family of indices draws from the one-dimensional parametric equivalence scale literature and aims to describe how much deprivation two demographically heterogeneous units with different needs must exhibit to be catalogued as equivalently deprived. Through the use of empirically developed counterfactual scenarios, we evaluate the effects on multidimensional deprivation incidence profiles of using the different approaches to measurement included in our proposed family of indices. The results of this evaluation demonstrate that neglecting differences in needs yields biased multidimensional deprivation incidence profiles. These results also shed light on the ability of our proposed measures to effectively capture differences in needs. Our family of measures is also evaluated in this paper in terms of its properties. These results prove our proposed approach to measurement has the proper orientation and is adequate for the purposes of poverty measurement.

Keywords: Multidimensional deprivation; poverty measurement; equivalence scales; heterogeneous households; individual heterogeneity

JEL Codes: D63, I32

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1 Introduction

There is an increasing interest in measuring poverty by assessing deprivation in multiple dimensions of well-being rather than by exclusively evaluating the ability to consume market commodities. Within this growing literature, most of the applications of multidimensional deprivation measurement use the Alkire and Foster (2011) method and either individuals or households as the unit of analysis[1].

However, differences in needs are present when measuring multidimensional deprivation across either individuals from different demographic population subgroups or households of different sizes and compositions. While pregnant women, for instance, need access to antenatal health services, school-age children need access to basic education services. Deprivation of antenatal health services is thus relevant only to pregnant women, and access to basic education services is only relevant to school-age children. Similarly, households without children are not necessarily deprived in the absence of educational services and vaccinations, just as households without pregnant women are not necessarily deprived because of a lack of antenatal health services. Differences in needs therefore, pose comparability challenges when measuring multidimensional deprivation across demographically heterogeneous units, such as households of different sizes and compositions or individuals of different age ranges and genders.

The individual-based applied multidimensional deprivation literature have addressed these differences in needs by restricting the analysis to demographically homogeneous individuals. For instance, in terms of the individual-based applied multidimensional literature, the majority of these studies focuses on measuring multidimensional deprivation among either children or adult populations[2]. To this date in our knowledge, no individual-based application of multidimensional deprivation measurement considering the whole age range of the population can be found.

In terms of household-based indices, policy influential applications as the global Multidimensional Poverty Index (MPI) launched by the United Nations Development Program (Alkire, Conconi, and Roche, 2013), the Colombian Multidimensional Poverty Index (Angulo, Díaz, and Pardo, 2015), or studies such as Alkire and Seth (2015), Alkire and Santos (2014), Ayuya, Gido, Bett, Lagat, Kahi, and Bauer (2015), Bader, Bieri, Wiesmann, and Heinimann (2015), Cavapozzi, Han, and Miniaci (2013), Mitra (2014), Alkire, Roche, Seth, and

[1] Although the terms 'multidimensional deprivation' and 'multidimensional poverty' are used interchangeably in literature, throughout this paper we use the former term to refer to indices that count the multiple deprivations jointly observed across a selected unit of analysis and, based on this counting procedure, identify the poor as the most deprived population. Examples of the long-standing literature in multidimensional deprivation measurement are the following studies: Townsend (1979), Atkinson and Bourguignon (1982), Mack, Lansley et al. (1985), Callan, Nolan, and Whelan (1993), Feres and Mancero (2001), Atkinson (2002), Alkire and Foster (2011), and Aaberge and Brandolini (2014a).

[2] Studies regarding child multidimensional deprivation are Roelen, Gassmann, and de Neubourg (2010), Roche (2013), Trani and Cannings (2013), Trani, Biggeri, and Mauro (2013), and Qi and Wu (2014). Studies that focus on multidimensional deprivation among an adult population include Oshio and Kan (2014) and Solaymani and Kari (2014).

Summer (2015), and Yu (2013), identify as most deprived those households that exhibit the largest number of dimensions in deprivation disregarding their size and composition. They therefore, either assume the same set of needs across households or ignore the fact that demographically dissimilar households have significantly different needs.

In contrast to the multidimensional measurement literature, a plethora of methods and techniques that account for differences in needs can be found in the one-dimensional welfare literature. Examples of such works include Kapteyn and Van Praag (1978); Pollak and Wales (1979); Blundell and Lewbel (1991); Coulter, Cowell, and Jenkins (1992a); Cowell and Mercader-Prats (1999); Duclos and Mercader-Prats (1999), and Ebert and Moyes (2003). They aim to provide societal profiles based on comparable household-based aggregates of income or expenditure obtained through the use of *equivalence scales*. Comparisons of multidimensional deprivation between demographically dissimilar units have yet to be described in the multidimensional measurement literature.

This paper proposes a family of indices that measures multidimensional deprivation across demographically heterogeneous units while explicitly taking into account differences in needs across them. The proposed approach extends the Alkire and Foster (2011) counting family of multidimensional poverty indices, providing a wider set of indices that aims to adjust for observable differences in needs across demographically heterogeneous units. This is the methodological contribution of this paper to the multidimensional measurement literature.

The choice of the individual or the household as the unit of analysis is not arbitrary. It involves a normative decision to be made during the multidimensional measurement process. Household-based measures conceive households as cooperative units that jointly face the deprivation suffered by the household members. Individual-based measures, in contrast, allow the unmasking of differences in multidimensional deprivation across demographic population subgroups, such as the case of gender differences analysed by Vijaya, Lahoti, and Swaminathan (2014) for Karnataka, India or by Agbodji, Batana, and Ouedraogo (2013) for Burkina Faso and Togo.

The family of indices that we propose in this paper allows multidimensional deprivation to be measured using either individuals or households as the unit of analysis. The choice of individual or household is therefore open to be made according to the context of each application. In the case of household-based multidimensional measures, the purpose of accounting for differences in needs is to enable pairs of households and thus different populations of households to be compared on a more equivalent basis. Similarly, in the individual-based case, the indices proposed in this paper aim to enable multidimensional deprivation comparisons of any two individuals and hence of different populations of individuals.

Furthermore, our proposed family of measures allows describing—under equivalent normative considerations—the burden that multidimensional deprivation places on each unit of analysis (either households or individuals). Such burden is expressed through a family of measures that contains absolute, relative and intermediate normative perspectives. Multidimensional deprivation is therefore described via count-based, share-based, or a mixture of count-based and share-based approaches to measurement.

To evaluate the effect of these different approaches to measurement of multidimensional deprivation incidence profiles, we construct counterfactual scenarios using the 2013 Paraguayan household survey. The obtained results demonstrate that neglecting differences in needs yields multidimensional deprivation incidence profiles to reflect not only differences in deprivation, but also differences in needs that should be tackled by the measurement process. Failure to take differences in needs into account, as the dimensions count-based approach to measurement proposed by the Alkire and Foster (2011) methodology, was found to cause biased multidimensional incidence profiles. These results also shed light on the ability of our proposed measures to effectively capture these differences in need.

In this paper we also evaluate our proposed family of measures through the properties that make it admissible for the purposes of multidimensional deprivation measurement. The result of this evaluation shows that our family of measures fulfils this purpose, has the proper orientation and is non-sensitive to non-relevant aspects of the distribution.

Our paper is organised as follows. The starting point is the background literature that analyses welfare comparisons in the presence of heterogeneous needs. Section 2 presents an overview of this literature and the equivalence scale notion that seeds the family of indices proposed in this paper. Then, in Section 3 we describe the formal setting that includes the Alkire & Foster family of indices, as background methodology, and the proposed family of multidimensional deprivation indices of this study. Sections 4 and 5 evaluate our measures, first disentangling how much of the differences in multidimensional deprivation incidence profiles are observed because unaddressed differences in needs, and then in terms of their proposed poverty measurement properties. In Section 6 we discuss and provide guidelines to set the most important context specific definitions that our proposed approach requires, and we conclude our study in Section 7 with some remarks.

2 Background

2.1 Welfare comparisons in the presence of heterogeneous needs

A variety of methods and techniques from the one-dimensional literature assess welfare and inequality rankings while taking into account differences in needs between households. Examples of such methods and techniques can be found in Kapteyn and Van Praag (1978); Pollak and Wales (1979); Blundell and Lewbel (1991); Coulter et al. (1992a); Cowell and Mercader-Prats (1999); Duclos and Mercader-Prats (1999), and Ebert and Moyes (2003). Within this literature, these technologies are known as *equivalence scales*. Their relevance is crucial for inequality and social welfare comparisons; as pointed out in Cowell and Mercader-Prats (1999, page 409): “Equivalence scales, by providing an interpersonally comparable measure of living standards, play a central role in the assessment of social welfare and income inequality. Failure to take account of the relationship between nominal and equivalised income can give a biased picture of both inequality and social welfare”.

In general, equivalence scales have been used to allow the construction of societal measures of welfare and inequality based on comparable household measurements of income or expenditure (Fisher, 1987; Muellbauer, 1974). These scales intend to reflect the amount of income required for households of different sizes and compositions to have the same welfare level (Pollak and Wales, 1979; Nelson, 1993). An important emerging fact from this literature is that there is no universally correct equivalence scale. Different procedures are justified according to different circumstances.

One particular branch of this literature has focused on providing a measurement approach to characterize the household income via a set of parameters which describe the responsiveness of the income with respect to the size of household needs. Measures developed under this perspective are known as *parametric equivalence scales* and are based on the selection of parametric values to typify this relationship in terms of the size and composition of the household. Examples of these technologies can be found in Atkinson and Bourguignon (1987); Buhmann, Rainwater, Schmaus, and Smeeding (1988); Coulter et al. (1992a), and Cowell and Mercader-Prats (1999). A general approach of this type of equivalence scale is analysed by Buhmann et al. (1988) and Coulter et al. (1992a), in which they express the y_h-household adjusted income as a function of the x_h-observed household income, the q_h-size of the household and a θ-scale relativity parameter. There, the household adjusted income is defined by

$$y_h = \frac{x_h}{(q_h)^\theta}. \tag{1}$$

In this approach, needs are expressed in terms of the size of the household, and the θ-scale relativity parameter varies from no adjustment of the household income by needs ($\theta = 0$) to a complete adjustment portrayed by the per capita household income ($\theta = 1$).

The proposed methodology of this paper draws from this one-dimensional parametric equivalence scale literature. Similarly to the equivalence scale of (1), the family of measures of this paper uses a scale relativity parameter θ to emphasise needs. The proposed methodology of multidimensional deprivation measurement of this paper aims to describe how much deprivation demographically heterogeneous units must exhibit to be catalogued as equivalently deprived. It enhances multidimensional deprivation comparisons across either households of different sizes and compositions or individuals from different demographic population subgroups and allows societal multidimensional indices based on more comparable profiles than those available in the applied multidimensional measurement literature.

2.2 Multidimensional deprivation measurement

Several conceptual approaches exist to measure welfare, and each chooses its specific focus: resources (income or others), basic needs, Sen's functionings or capabilities (Sen, 1993), rights, happiness and so on. As such, the conceptual focus of any index and the selection of dimensions and indicators correspond to a normative selection to be taken for each specific context. The family of multidimensional measures of this paper is proposed to be applied by any of these different conceptual approaches.

Within the multidimensional literature, two alternative procedures identify the poor population and aggregating dimensions: the 'welfare approach' and the 'counting approach'. The former approach combines several dimensions into a single variable and sets a threshold to differentiate between poor and non-poor populations. The welfare approach has been studied by Bourguignon and Chakravarty (2003), Seth (2009), and Seth (2010), among others.

By contrast, the counting approach—as its name indicates—counts the number of dimensions in which persons suffers deprivation, and the identification of the poor person relies on defining how many dimensions must be deprived for someone to be categorized as multidimensionally deprived. Examples of these types of measures and analyses are proposed by Townsend (1979), Atkinson and Bourguignon (1982), Mack et al. (1985), Callan et al. (1993), Feres and Mancero (2001), Atkinson (2002), Aaberge and Brandolini (2014a), and Alkire and Foster (2011). Efforts have been made within the literature to analyse both approaches under a common framework (for a such study see Atkinson (2003)). However, as pointed out by Aaberge and Brandolini (2014b), this discussion is still inconclusive.

The family of measures proposed in this paper stands, specifically, within the counting multidimensional deprivation literature and extends the Alkire and Foster (2011)'s methodology. For brevity, we henceforth refer to the multidimensional poverty measurement method proposed by Alkire and Foster (2011) using the abbreviation 'AF' or 'AF methodology'.

3 Formal setting

This section presents the formal setting that frames the proposed family of multidimensional deprivation indices of this article. This setting contains the most commonly used measures of the AF methodology along the measures that we propose as extensions of this methodology. The section first describes the AF multidimensional deprivation measurement methodology while using a slightly modified notation; and subsequently, presents the proposed extension of this paper for such methodology.

3.1 The AF methodology

Consider a population consisting of $I \geq 1$ individuals evaluated across $J \geq 2$ indicators. The AF method begins by defining an $I \times J$ matrix $\mathbf{A} = [a_{ij}]$, where each row corresponds to an individual and each column to an indicator quantifying the individuals' achievements such as education level, nutrition, health status, et.c. Obviously, greater values of an achievement indicator refer to better-off conditions, and lower values of it refer to worse-off conditions.

Each column of $\mathbf{A}$ is either a cardinal or an ordinal achievement indicator[3]. More precisely, the cell a_{ij} of the matrix $\mathbf{A}$ quantifies the the j achievement of i individual. The domain of the $\mathbf{A}$ matrix is the non-negative orthant of the J-dimensional Euclidean space. The AF methodology defines the i individual as deprived in the j indicator by placing a threshold $z_j > 0$ for each $j = 1, \ldots, J$. As such, the $\mathbf{z}$-vector of non-negative thresholds $\mathbf{z} = (z_1, \ldots, z_J)$ is defined and the i individual is said to be j-deprived if and only if $a_{ij} < z_j$.

The breadth of the suffered deprivation g_{ij}^{α} is given by:

$$g_{ij}^{\alpha} = \begin{cases} \left(\dfrac{z_j - a_{ij}}{z_j}\right)^{\alpha} & \text{if } a_{ij} < z_j \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where $\alpha \geq 0$ is the poverty aversion parameter. The α parameter, first introduced in the poverty measurement literature by Foster, Greer, and Thorbecke (1984) and used by Alkire and Foster (2011), assigns greater emphasis to the most deprived or lowest achieving individuals. The greater the value of α, the larger the accentuation of g_{ij}^{α} on the most deprived.

However, if the achievement variable is ordinal, the g_{ij}^{α} expression is valid only for $\alpha = 0$, and g_{ij}^{0} takes the value of either 1 or 0, indicating the presence or absence of deprivation. Hence, as discussed by Alkire and Foster (2011), the breadth of the suffered deprivation with $\alpha > 0$ can be defined only for cardinal indicators. Given that most of the public policy indicators in current use are ordinal, we restrict g_{ij}^{α} strictly to the case of $\alpha = 0$; and henceforth is denoted as g_{ij}.

The AF methodology continues then by aggregating deprivations across dimensions for each i individual with a c_i metric:

$$c_i = \sum_{j=1}^{J} g_{ij}. \tag{3}$$

[3]A cardinal indicator is such that any of its values measures the size of the achievement. This means that the comparison between any two given observed points of a cardinal indicator can be commensurate with the difference between their respective sizes. For example, years of education is a cardinal achievement indicator because having two years of education is twice the number of one year of education. In contrast, an ordinal indicator does not allow measuring the size of the achievement, but rather only indicates a particular ordering between situations. An example of an ordinal achievement indicator is the self-assessment of health status, which takes the categories of "very poor", "poor", "good", and "very good". Note that in this case, we are unable to evaluate the 'size' of the situation. For instance, if we compare two observations, one person having very good health and another person having very poor health, we do not observe the size of the difference between the two situations. In this latter case, we only know that the first person has better off self-assessed health status than the second one, but we do not know the magnitude of the difference in self-assessed health status between the two persons.

Subsequently, a threshold k to identify the multidimensionally deprived is placed over this c_i metric. As a result, any i individual satisfying $c_i \geq k$ is identified as multidimensionally deprived. Then, each g_{ij} is censored to zero in case the i individual is identified as not multidimensionally deprived. Therefore, $g_{ij}(k) = 0$ for any i individual that satisfies $c_i < k$.

In order to obtain societal metrics, the simplest measure that AF proposes is the H-multidimensional deprivation incidence. This first metric corresponds to the proportion of people identified as multidimensionally deprived using the k-threshold. The second most important societal metric that AF proposes and that is currently in use by most of the applications of the method is the M_0-adjusted headcount ratio. AF defines the adjusted headcount ratio as $M_0 = \mu(g_{ij}(k))$, where $\mu(g_{ij}(k))$ corresponds to the average $g_{ij}(k)$ for $i = 1, 2, \ldots, I$ and $j = 1, 2, \ldots, J$, i.e.

$$\mu(g_{ij}(k)) = \frac{1}{I \cdot J} \sum_{i=1}^{I} \sum_{j=1}^{J} g_{ij}(k).$$

However, when measuring deprivation, demographic heterogeneity plays a central role in the definition of what can be considered a lack of a minimum achievement. Children, for instance, can be considered deprived when they are not accessing basic education services, unlike adults, who can be considered deprived in the same education dimension when they do not know how to read and write. As another example, while adult populations that do not have access to job opportunities despite seeking those can be defined as deprived in employment, children cannot be defined as deprived in the absence of employment. Then, when it comes to measuring multidimensional deprivation, differences in needs reflected by the different populations where each indicator is applicable to be measured, bring comparability challenges to measuring how many dimensions in deprivation a particular individual or household might exhibit to be catalogued as multidimensionally deprived.

Note that the theoretically developed AF methodology does not address the comparability problems that differences in needs might bring to multidimensional deprivation measurement. It evaluates each i individual of the population across the all J achievements. Therefore, it implictly assumes that all individuals of the population as exhibiting the same set of needs.

The applied multidimensional deprivation literature have addressed these differences in needs by restricting individual-based measures of multidimensional deprivation to the analysis of demographically homogeneous individuals. Examples of this approach are studies that focus on measuring multidimensional deprivation among either children or adult populations such as Roelen et al. (2010), Roche (2013), Trani and Cannings (2013), Trani et al. (2013), Qi and Wu (2014), Oshio and Kan (2014) and Solaymani and Kari (2014).

In terms of household-based indices, no family of measures have been formally developed using such unit of analysis. Still, influential policy applications as the global MPI launched by the United Nations Development Program

(Alkire et al., 2013) and the Colombian Multidimensional Poverty Index (Angulo et al., 2015), or studies such as Alkire and Seth (2015), Alkire and Santos (2014), Ayuya et al. (2015), Bader et al. (2015), Cavapozzi et al. (2013), Mitra (2014), Alkire et al. (2015), and Yu (2013), they all use household as the unit of analysis and identify as most deprived those that exhibit the largest number of dimensions in deprivation disregarding their size and composition. They therefore either assume the same set of needs across households or do not consider the fact that demographically dissimilar households have significantly different needs.

The proposed method of this paper enables the measurement of multidimensional deprivation across demographically heterogeneous units (i.e., households of different sizes and compositions or individuals from different demographic population subgroups) while taking into account observable differences in need. Subsection 3.2 below describes the proposed methodology.

3.2 The proposed family of multidimensional deprivation indices

The proposed methodology of this paper begins by defining for each j achievement the sub-population group for which it is relevant to be measured. We call this the *applicable population subgroup* for achievement j, and we will measure the presence or absence of the j deprivation only within this set of sample units. This feature of our methodology captures individual differences in needs, corresponding to the traditional approach in the policy context to tracking indicators.[4] With this feature, we bridge the gap between theoretically developed multidimensional indices and policy-oriented single indicators design.

This feature is formalized with an $I \times J$ matrix of applicable population subgroups that we call $\mathbf{S}$. There are as many as J applicable population subgroups, and any two applicable population subgroups are not necessarily mutually exclusive. The cell s_{ij} of the matrix $\mathbf{S}$ is an indicator variable that takes the value of 1 if and only if the i individual belongs to the applicable population subgroup of the j achievement, and 0 otherwise. For instance, according to the Millennium Development Goals, access to primary education is relevant to be measured among school-age children; thus, cell s_{ij} takes a value of 1 whenever the i-individual is aged 6 to 15 years old and 0 otherwise.

Any observed j achievement for the i person that does not belong to the applicable population of such achievement is, therefore, defined as unimportant for the measurement process. Thus, the g_{ij} individual dimensional deprivation

[4] A long-standing tradition of policy indicators evaluates deprivation for each particular achievement over a specific sub-population of interest. For instance, one of the Millennium Development Goals (MDG) launched by the United Nations Development Programme and adopted by several countries to be achieved by 2015 is universal primary education. Another MDG is universal access to reproductive health. Both access to primary education and access to reproductive health services are relevant for measurement only among their particular applicable populations, which are children 6 to 15 years of age and pregnant women, respectively.

indicator evaluated on its applicable population is denoted by $g_{ij}(s_j)$ and given by:

$$g_{ij}(s_j) = \begin{cases} 1 & \text{if } a_{ij} < z_j \text{ and } s_{ij} = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

where s_j denotes the applicable population of the j achievement. Then, we define the $I \times J$ matrix $\mathbf{G} = [g_{ij}(s_j)]$, where each row of $\mathbf{G}$ corresponds to an individual and each column to a binary indicator of presence or absence of deprivation for the i-individual in the j-indicator. As such, $\mathbf{G}$ is a binary matrix.

We continue describing the proposed method to commensurate the burden that multidimensional deprivation places on the household, whenever household is selected as the unit of analysis. Later in this manuscript—as an extension of household-based measures—the individual-based method is presented.

Household-based metrics

Assume that each i individual belongs to a particular h household, and each household contains q_h household members. The d^{β}_{hj}-dimensional deprivation indicator for the h household and the j dimension is defined as:

$$d^{\beta}_{hj} = \begin{cases} \left(\sum_{i=1}^{q_h} g_{ij}(s_j) \right)^{\beta} & \text{if } \sum_{i=1}^{q_h} g_{ij}(s_j) > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

where $\beta \in \{0, 1\}$ is the parameter of aversion to deprivation. Whenever $\beta = 0$, then household dimensional deprivation is expressed by a $\{0, 1\}$ indicator of absence or presence of at least one j deprived household-member. On the other hand, if $\beta = 1$, then the dimensional deprivation is expressed by the count of deprived household members in the j dimension.

The β parameter of aversion to deprivation is analogous to the α parameter of poverty aversion introduced by Foster et al. (1984) and used by Alkire and Foster (2011) to assign increasing value to the most deprived dimensions. Similarly to the α parameter of AF method, whenever $\beta = 0$, dimensional deprivation is expressed as an indicator of presence or absence of deprivation in the j dimension. However, while in the AF method $\alpha > 0$ can be used only in case that the j dimension is captured by a cardinal achievement indicator, for $\beta = 1$ commensurates the household deprivation breadth in the j dimension, without necessarily enforcing the use of cardinal achievement indicators and in terms of the number of j deprived household-members.

Due to the ordinal nature of most of policy indicators, current household-based applications of AF method have been restricted to measure the burden that dimensional deprivation places on the household by indicating the presence or absence of at least one household member under deprivation in this dimension. This particular approach corresponds to using d^0_{hj} to express dimensional deprivation, which is setting $\beta = 0$.

The use of $\beta = 0$, however, does not allow household metrics to be sensitive to increments in the number of deprived persons in an already deprived dimension. For instance, when evaluating access to primary education, a household with two school-aged children, one child attending school and the other not attending, registers $d^0_{hj} = 1$. Now, if this same household, as a result of a deprivation increment, increases its number of children who are not attending school to two, its d^0_{hj} indicator remains invariant.

In contrast, our proposed methodology enables expressing household dimensional deprivation with $\beta = 1$, which produces a measure of dimensional deprivation that is sensitive to increments in the number of deprived persons in already deprived dimensions. For instance, in the example of the previous paragraph, if we evaluate school attendance in the household with one deprived school-age child, then $d^1_{hj} = 1$. But if the household has two children deprived of school attendance, then $d^1_{hj} = 2$: a value that is twice as large that of the initial case. We further discuss and illustrate this advantage of the proposed method in Section 5.

Of course, not every household has the same set of dimensional needs. In fact, the number of j applicable household members generally varies across households. To account for this, we define n^β_{hj} to be the size of the h household needs on the j dimension:

$$n^\beta_{hj} = \begin{cases} \left(\sum_{i=1}^{q_h} s_{ij}\right)^\beta & \text{if } \sum_{i=1}^{q_h} s_{ij} > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

where n^0_{hj} measures the size of household j dimensional needs through an indicator of whether or not the household has need in the j dimension (i.e., has at least one household member that could suffer deprivation in such dimension). In contrast, the n^1_{hj} size of needs informs the number of household members that exhibit need in the j dimension. For instance, in our same example of school attendance, since the h household has two school-age children, then we know that $n^0_{hj} = 1$ and $n^1_{hj} = 2$.

The second stage of the proposed methodology of this paper consists of measuring the burden that multidimensional deprivation places on the household via a functional form that enables capturing either count-based, shared-based or a mixture of these two approaches to measurement. In this vein, following Cowell and Mercader-Prats (1999) and Buhmann et al. (1988), and using the equivalence scale presented in equation (1), we express the burden of multidimensional deprivation as:

$$m^{\beta,\theta}_h = \begin{cases} \dfrac{\sum_{j=1}^J d^\beta_{hj}}{\left(\sum_{j=1}^J n^\beta_{hj}\right)^\theta} & \text{if } \sum_{j=1}^J n^\beta_{hj} > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where $\theta \in [0,1]$ is a deprivation response scale parameter that reflects the relativity of the response of the burden of deprivation to the scale of household needs. In the case that $\theta = 0$, we are in the presence of a count-based approach, and no discounting in needs is applied at all. Thus, the household is assumed as not receiving any advantage from the cooperative unit, and therefore the burden that deprivation places on the household is not lightened to any degree from the scale of the needs.

On the other hand, when $\theta = 1$, we are using a share-based approach. While the count-based structure places greater emphasis on larger households without accounting for any possible scale economy that might arise at this level, the share-based approach places greater emphasis on small households because they are more prone to registering the maximum possible burden of deprivation. Values of θ different than 0 or 1, aim to describe $m^{\beta,\theta}$ through an intermediate approach that lie in between of count-based and share-based perspectives.

Current household-based indices that use the AF method measure the burden of multidimensional deprivation through the $m^{0,0}$ metric, which corresponds to counting the number of dimensions in deprivation. However, the $m^{0,0}$ metric does not discount by household needs at all and thus, does not differentiate the deprivation burden of non-deprived and non-applicable dimensions. This induces the observation of a systematically lower burden of multidimensional deprivation across small and demographically homogeneous households, as we shall see in Section 4.

In contrast, our proposed $m_h^{\beta,\theta}$ family of measures allows not only a count-based approach to measurement, which corresponds to setting $\theta = 0$; but also takes into account heterogeneous household needs within and across dimensions through using any $0 < \theta \leq 1$. In fact, whenever θ is set in such interval, the burden of household multidimensional deprivation is discounted by the household needs and takes into account the scale advantages that the household receives to lighten the burden that deprivation places on it.

The four most intuitive measures of the burden that multidimensional deprivation places on the household are those which set $\theta \in [0,1]$ and $\beta \in \{0,1\}$. They are termed as follows:

- The dimensions count-based approach to measurement: $m^{0,0}$
- The dimension share-based approach: $m^{0,1}$
- The deprivations count-based approach: $m^{1,0}$
- The deprivations share-based approach: $m^{1,1}$.

Henceforth, these measures are referred interchangeably in terms of their names or their respective $m^{\beta,\theta}$ notations.

Identification of the multidimensionally deprived

For a given combination of β and θ, households exhibiting at least a k burden of multidimensional deprivation are identified as the multidimensionally deprived.

Parameter k represents the multidimensional deprivation threshold above of which the most deprived household are observed. The k threshold takes values between zero and the maximum possible observable $m_h^{\beta,\theta}$. The plausible k needs to be defined according to the context of each application.

In conjunction with the use of the k threshold, a p_h-binary indicator of presence or absence of multidimensional deprivation, naturally arises as follows:

$$p_h = \begin{cases} 1 & \text{if } m_h^{\beta,\theta} \geq k \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

While applications of the AF method sort households under the basis of $m^{0,0}$ and households satisfying $m_h^{0,0} \geq k$ are identified as the multidimensionally deprived, the proposed methodology of this paper enables the identification of the most deprived to be done under the basis of any $m^{\beta,\theta}$. The implications on multidimensional deprivation incidence profiles of using different $m^{\beta,\theta}$ measures to identify the multidimensionally deprived are investigated and discussed in Section 4. We continue presenting the proposed methodology for aggregating household multidimensional deprivation at the society level.

The family of societal measures

Let R denoting the total number of households. Then, the simplest metric to represent the overall society multidimensional deprivation incidence is:

$$H(m^{\beta,\theta}) = \mu(p_h), \tag{9}$$

where $\mu(p_h)$ corresponds to the average value of p_h for $h = 1, \ldots, R$. Throughout this paper, $H(m^{\beta,\theta})$ corresponds to the proportion of multidimensionally deprived population identified on the basis of a particular $m_h^{\beta,\theta}$ metric. Note that the Alkire and Foster H-headcount ratio corresponds here to the case where $m_h^{0,0}$ is used to identify the most deprived population, which is $H(m^{0,0})$.

Now, to construct societal metrics of the average burden that multidimensional deprivation places across households, we censor to zero any $m_h^{\beta,\theta}$ for non-multidimensionally deprived household, namely, $m_h^{\beta,\theta} = 0$ for all h such that $p_h = 0$. Then, the household burden of multidimensional deprivation after the identification of the multidimensionally deprived is denoted as $m^{\beta,\theta}(k)$ and the societal mean burden of multidimensional deprivation is defined as:

$$MD^{\beta,\theta} = \mu(m^{\beta,\theta}(k)), \tag{10}$$

where $\mu(m^{\beta,\theta}(k))$ corresponds to the average value of $m^{\beta,\theta}(k)$ for $k = 1, 2, \ldots, R$. In comparison to the M_α family of measures of the AF method, our $MD^{0,0}$ metric corresponds to the M_0 metric of the Alkire and Foster (2011) method. As such, the proposed $MD^{\beta,\theta}$ family of measures constitutes a broader set of metrics that takes into account count-based, share-based and intermediate approaches to measure the burden that multidimensional deprivation places on the household.

In general, given the ordinal nature of policy indicators, most current applications on the Alkire and Foster (2011) method are able to describe societal multidimensional deprivation through $H(m^{0,0})$ and $MD^{0,0}$. Our proposed approach, in contrast, allows describing multidimensional deprivation in terms of any $H(m^{\beta,\theta})$ and $MD^{\beta,\theta}$ with $\beta \in \{0,1\}$ and $\theta \in [0,1]$.

Weights

For completeness purposes and to guide applications where indicators have different relative importance across each-other, this section introduces and describes a weighting system to differentiate these relative importances.

Consider the $\mathbf{w} = (w_1, w_2, \ldots, w_J)$ vector of non-negative importance weights, where $w_j \geq 0$ denotes the relative importance weight for the j achievement in the overall deprivation evaluation, and satisfies $\sum_{j=1}^{J} w_j = 1$. This weighting system can be used to aggregate deprivations across the J dimensions and obtain the burden of multidimensional deprivation as:

$$\mathbb{m}_{\mathbb{h}}^{\beta,\theta} = \begin{cases} \dfrac{\sum_{j=1}^{J} w_j d_{hj}^{\beta}}{\left(\sum_{j=1}^{J} w_j n_{hj}^{\beta}\right)^{\theta}} & \text{if } \displaystyle\sum_{j=1}^{J} w_j n_{hj}^{\beta} > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

This $\mathbb{m}_{\mathbb{h}}^{\beta,\theta}$ burden of multidimensional deprivation represents the $\mathbf{w}$ scaled variant of equation (7). The application of the $\mathbf{w}$ dimensional weights produces, subsequently, societal measures H and $MD^{\beta,\theta}$ to be updated using this $\mathbf{w}$ scaled variant of $m_h^{\beta,\theta}$.

The selection of these dimensional weights can be devised according to the purpose of the measure and by different alternative procedures such as normative selection or data-driven techniques. For a study of alternatives to setting weights in a multidimensional index, see Decancq and Lugo (2013).

The individual-based method

The methodology introduced here is proposed to enable individual-based multidimensional deprivation measurement in presence of different needs across demographically heterogeneous population subgroups. In the context of this paper, this approach is named as the individual-based method, as it is derived as a special case of the previously described household-based measures. In this proposed individual-based method each household in the society is assumed as consisting of one member, which simply implies each person is in its own household. The afore-presented household-based measures are consequently derived. Hence, both the dimensional deprivation indicator and the burden of multidimensional deprivation are obtained without aggregating at the household level.

We remark that the d_{hj}^{β}-dimensional deprivation indicator for the h household in the j dimension was developed as an aggregation of the household members' $g_{ij}(s_j)$ individual deprivation indicators to the power of β (see (5)). However, this aggregation and the β parameter have no relevance in an individual-based method because in this case the resulting measure is always a binary variable of presence or absence of deprivation. Consequently, the $m_h^{\beta,\theta}$-burden of multidimensional deprivation for the h household, becomes also non-sensitive to different values of β and expressed independently for each i individual. We denote this variant of (7) as m_i^{θ}.

Still, the use of the θ parameter in the individual-based method expresses the responsiveness of deprivation to the size of the individual's needs. Similarly to the household-based case, in the individual-based method, the use of the θ parameter allows expression of the multidimensional deprivation burden that the i individual suffers, either as a count of dimensions on deprivation, or as a proportion of dimensions of deprivations or as a mixture of these two types of measures.

The use of this individual-based method naturally produces an identification of the most deprived individuals with any m_i^{θ} measure and defining as multidimensionally deprived those satisfying $m_i^{\theta} \geq k$. The societal measures H and MD, are therefore, developed using the individual-based variants of the measures.

We point out that the individual-based proposed approach when setting $\theta = 0$ corresponds to the individual-based AF methodology. In this case the proportion of multidimensionally deprived individuals is expressed by $H(m_i^0)$ and the MD^0 metric results equivalent to the AF metric M_0.

We now proceed to evaluate the implications of using different possible measures to identify the multidimensionally deprived population.

4 Whom are identified as multidimensionally deprived across measures?

In this section, we evaluate empirically the effects on multidimensional deprivation profiles of using the AF's *dimensions count-based* approach to measurement and compare it with those obtained using other members of the family of measures that we propose in this paper. The analysis is carried out making use of the data that is presented in the next section.

4.1 Data

For the empirical analysis in this paper, a multidimensional deprivation index is built using the 2013 Paraguayan Household Survey (PHS). The PHS is a cross-sectional living conditions survey that has been collected yearly since 1984 by the Paraguayan National Statistical Department. Referred to as the *Encuesta Permanente de Hogares*. It uses a two-stage, clustered probabilistic sample design that was stratified in the first stage by 31 geographical domains.[5]

[5]The strata corresponded to rural and urban areas of 15 out of the total 17 Paraguayan counties (*departamentos*) and the national capital of Asunción.

The PHS 2013 captures a broad range of living condition indicators. It provides national estimates for income poverty, inequality, and some key quality of life descriptors. The questionnaire of the PHS 2013 includes information regarding education, health, the labour market, individual income, dwelling conditions, and international migration and a special module for agriculture and forestry activities.

Table 1 describes the items included within the multidimensional index of deprivation constructed for the analysis purposes of this paper. In particular, this index example captures information on access to health services, education, and dwelling conditions, across five deprivation indicators: health insurance non-

Table 1: Example of multidimensional indicator: Dimensions, indicators, weights, applicable population subgroups and deprivation criteria

Dimension	Deprivation indicator	Applicable population subgroups where the indicator is relevant to be measured	A person from the applicable population is deprived if:
Access to health services	Health insurance non-coverage	Any person	Is not covered by any health insurance.
	Non-access to health services when needed	Any person that was sick or had an accident during the 90 days previous to the interview	Did not receive institutional care*.
Education	Non-school attendance	5 - 17 years old population	Is not attending school.
	Low educational achievement	Population 18 years old and over	Has less than 9 years of completed education.
Dwelling conditions	Sub-standard housing	Any person	Lacks at least 2 of the following 3 dwelling conditions: flooring different from earth or sand; adequate material of ceilings**; and adequate material of walls***.

Notes: *Institutional care corresponds to attention received by a professional health worker (physicist, nurse, dentist or professional midwife) in private or public health institution (It is not a health care institution: pharmacy, empirical medicine man store, own house, other's house). **Inadequate ceiling material refers to the following: Straw, eternit, clapboard, palm trunk, cardboard, rubber, packaging timber, other. ***Inadequate wall materials refer to the following: wattle, mud, wood, palm trunk, cardboard, rubber, wood, another material, or no wall at all.

coverage, non-access to health services, non-school attendance, low educational achievement, and substandard housing. The substandard housing indicator was included in this Paraguayan index to illustrate how a deprivation indicator invariant across household members may be incorporated into an index that uses the proposed methodology of this paper. As such, the applicable population of the substandard housing indicator corresponds to any household member and each individual is defined as deprived whenever the housing lacks from at least 2 of the 3 considered dwelling conditions (flooring different from earth or sand, adequate material of ceilings, and adequate material of walls).

In 2013, the PHS was collected from a sample of 21,207 persons across 5,424 households. Of the 21,207 interviewed individuals for PHS 2013, 264 observations were excluded from the analysis because they correspond to individuals that do not belong to the household unit (i.e., domestic personnel), and 34 observations were also excluded because of non-response in at least one of the five considered indicators. Thus, the effective sample comprises 20,909 interviewed persons across 5,423 households.

4.2 Observed multidimensional deprivation incidence profiles

Table 2 presents the proportion of households with at least one deprived household member in each of the five considered indicators by household size. This corresponds to the mean d^0_{hj}-dimensional deprivation indicator across the 5,423 observed Paraguayan households. Reading the table by lines, it can be seen that larger households exhibit a larger proportion of dimensional deprivation than smaller households. The dimensions more prone to this behaviour are health insurance non-coverage, non-access to health services, non-school attendance, and low educational achievement. A positive relation between household size and dimensional deprivation is observed because the number of persons in the applicable population increases as the household size increases. Take for example the non-school attendance indicator in Table 2, which is applicable for children 5 to 17 years of age. One-person households are rarely composed of this population subgroup because school-age children cannot form a household. Therefore, the proportion of households consisting of one person that are dimensionally deprived in school attendance is 0%. Conversely, 21.4% of households consisting of seven or more persons are deprived of school attendance because they contain on average four children.

If, subsequently, the dimensions exhibiting deprivation are counted and the multidimensionally deprived households are those with the largest count of these dimensions on deprivation, larger and more heterogeneous households tend to be identified as the most deprived. The following paragraphs elaborate further on this.

With the purpose of comparing the multidimensionally deprived population of households identified using different $m^{\beta,\theta}$ measures, households are sorted on the basis of each $m^{\beta,\theta}$ score and the first 40% most deprived (2,168 households) are identified as multidimensionally deprived. The population of house-

Table 2: Proportion of households with at least one deprived person from the applicable population (%)

	Persons per household							Total
	1	2	3	4	5	6	7 or more	
(1) Health insurance non-coverage	70.3	75.2	79.9	81.0	84.3	91.4	93.8	81.3
(2) No access to health services	12.5	17.7	15.1	19.7	20.1	23.6	29.6	19.0
(3) Non-school attendance	0.0	1.9	2.9	4.4	5.8	9.7	21.4	5.5
(4) Low educational achievement	61.4	64.8	57.8	65.4	68.6	78.3	88.3	67.0
(5) Sub-standard housing	25.5	25.0	18.5	19.9	23.5	24.9	34.2	23.3
Sample number								
Number of households	593	836	1,135	1,108	771	466	514	5,423
% of individuals	2.8	8.0	16.3	21.2	18.4	13.4	19.9	100

Source: author's calculations based on 2013 PHS.

holds identified as the most deprived using the dimensions count-based approach ($m^{0,0}$) is compared with regard to those obtained using the other three $m^{\beta,\theta}$ most representative of the proposed methodology: the dimension share-based approach ($m^{0,1}$), the deprivations count-based approach ($m^{1,0}$), and the deprivations share-based approach ($m^{1,1}$).

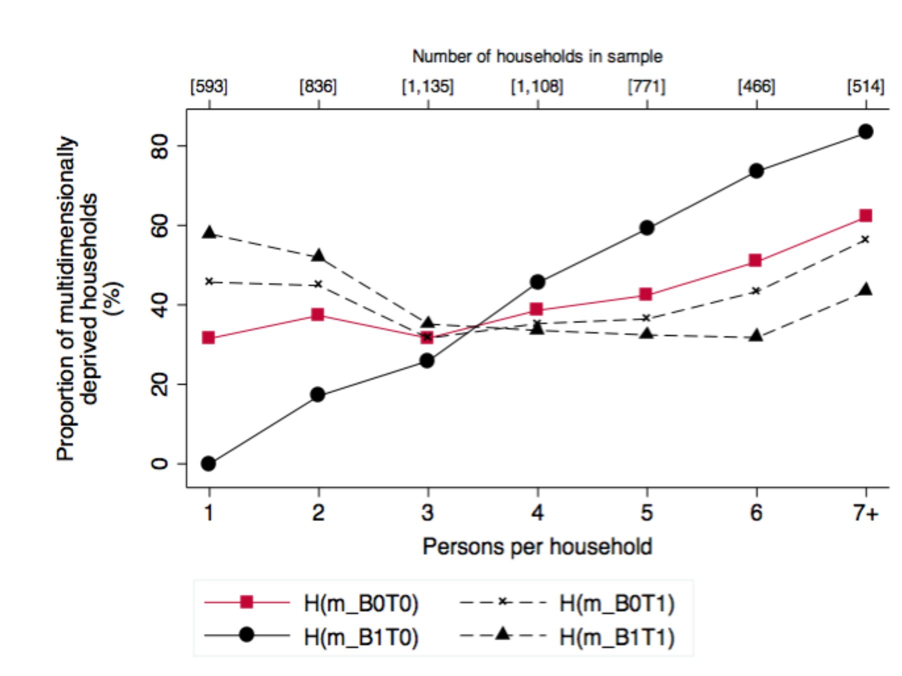
Note that identifying a fixed share of the population (40% in this case) as the most deprived is different from placing a particular k multidimensional threshold over the $m^{\beta,\theta}$ score. Given that the range of variability of $m^{\beta,\theta}$ varies along the β and θ parameters, the use of a fixed share of households enables us to compare the different deprived populations on an equal basis. The particular 40% share of households arose as a plausible natural breaking point in the distribution of deprivations observed by the multidimensional index in the analysis. Nonetheless, in Section 4.5 we test the robustness of the obtained results under other different possible shares of the population.

Figure 1 plots the obtained H-multidimensional deprivation incidence by household size for the four $m^{\beta,\theta}$. In the figure, the vertical axis corresponds to the proportion of households of each size identified as multidimensionally deprived. For example out of the total observed 514 households consisting of seven or more persons, about 80% of them are identified as multidimensionally deprived when a deprivations count-based approach ($m^{1,0}$) is used to sort households.

As expected, the results indicate that the H-multidimensional deprivation incidence varies across household size and measures. No adjustment by differences in needs corresponds to measures that use $\theta = 0$. The profiles obtained upon these bases show the greatest proportion of multidimensionally deprived among large households, as well as the lowest proportion among small households. In particular, when using the AF-proposed $m^{0,0}$, households consisting of seven or more persons register 29.2 percentage points (p.p.) larger multidimensional deprivation incidence than households consisting of one person.

Any $\theta > 0$ enables the burden of household multidimensional deprivation to be adjusted by household needs, increasing the amount of the adjustment as θ increases. Then, contrary to count-based approaches, a deprivations share-based approach (triangle markers in the figure $-m^{1,1}-$) produces 57.8% of households consisting of one person being catalogued as multidimensionally deprived and 43.6% of households consisting of seven or more persons being catalogued as multidimensionally deprived. Thus, in this case, a 14.2 p.p. higher incidence of multidimensional deprivation is observed among smaller households than across

Figure 1: Proportion of multidimensionally deprived households, $H(m^{\beta,\theta})$, across household size



Source: author's calculations based on 2013 PHS.

larger households.

These descriptive statistics suggest that identifying the most deprived on the basis of a household burden of multidimensional deprivation not adjusted by household needs results in greater H-deprivation incidence among larger households. Multidimensional deprivation incidence among larger households reduces as adjustment by the size of the needs increases. The use of different $m^{\beta,\theta}$ measures to sort households produces different profiles of multidimensional deprivation incidence, and these results are driven by the size of the household needs.

What should we make of these differences? On one hand—as particular studies from the one-dimensional equivalence scale literature suggest—one could argue that there is no correct or incorrect equivalence scale and that different measures are justified according to different circumstances (Cowell and Mercader-Prats, 1999, pg.409). In this vein, the selection of the measure to describe household multidimensional deprivation constitutes a context-specific normative definition. While count-based approaches ($\theta = 0$) give either to each dimension (using $\beta = 0$) or to each deprivation ($\beta = 1$) an equal absolute value in the measurement of the burden of multidimensional deprivation, deprivation share-based approaches ($\theta = 1$) give an equal absolute value to each household, disregarding its demographic composition and taking into account the scale economies that arise at this level.

An intermediate normative perspective corresponds to setting the θ parameter between these two solutions. The value of θ reflects the responsiveness of the burden of deprivation to the scale of needs; values of θ close to zero convey a lower response of the burden of multidimensional deprivation to the size of the needs. Conversely, values of θ close to one convey a greater response of the burden of deprivation to the size of the needs.

On the other hand, researchers can consider—as we do in this paper—differences in need as a 'legitimate' source of variation in the observed multidimensional deprivation profiles that should be tackled by the measurement process. Following the framework set up by Fleurbaey (2007) in social choice on equity, responsibility, and fairness, and in particular the proposed approach of Fleurbaey and Schokkaert (2009) to analyse fair and unfair health and healthcare inequalities, differences in achievement levels (such as health or educational attainment) are considered as caused by myriad factors, some of which can be catalogued as producing fair/legitimate differences and others as producing unfair/illegitimate differences. In particular, for the case of health and healthcare inequalities, Fleurbaey and Schokkaert (2009) defined as legitimate or fair those differences attributed to causes that fall under individuals' responsibility. Legitimate differences in this context correspond, therefore, to those derived from preferences.

In light of this framework, one can argue differences in multidimensional deprivation measurements should not arise from legitimate causes and we should therefore in our methodologies account for the differences that needs bring, as well as for any of other legitimate causes, such as differences in preferences. In this paper, for the sake of simplicity and as a first effort in the literature to account for differences in needs, we focus on accounting strictly for them. The effect that other sources of fair/legitimate differences, such as preferences, could have over multidimensional deprivation incidence profiles is left for further research. Analysis of the relation between multidimensional poverty and preferences can be found in Decancq, Fleurbaey, and Maniquet (2014).

Therefore, we evaluate how effectively each of the $m^{\beta,\theta}$ measures accounts for differences in needs. The methodology for approaching such evaluation and the results are presented in the following sections.

4.3 Method

To determine the ability of any multidimensional deprivation measure to account for differences in needs, we contemplate direct and indirect standardization techniques. As proposed by Fleurbaey and Schokkaert (2009), we consider both standardization techniques in light of their embedded ethical conditions and implications. As such, a desirability condition that resembles an indirect standardization technique is set out here to be attained by a multidimensional deprivation incidence profile. Based on this condition, we determine how much of the observed profile results from differences in needs and this regard its performance is evaluated. The next paragraph describes this condition.

Desirability condition. An *unbiased* multidimensional deprivation incidence profile is such that it is unable to distinguish between two population groups that have no systematic differences in deprivation between each other but only different sets of needs. As such, any two households in a household-based scenario or any two individuals in an individual-based scenario with no systematic difference in deprivation between the two of them must be classified equivalently as either multidimensionally deprived or non-multidimensionally deprived, regardless of the size of their needs.

Multidimensional deprivation incidence profiles that are unable to equivalently classify (as multidimensionally deprived or non-multidimensionally deprived) two households with differences in $m^{\beta,\theta}$, strictly caused by differences in needs, are said to provide a *biased* picture of societal multidimensional deprivation incidence.

If we can confirm that a particular $m^{\beta,\theta}$ measure is able to provide an equivalent measurement for any two households with no systematic differences in deprivation but only differences in needs, we also know that a multidimensional profile based on such an $m^{\beta,\theta}$ measure portrays differences in incidence that are not driven by differences in needs.[6]

[6] Another possible course of action could be using an alternative condition. This alternative condition can be set out in light of a direct standardization procedure. It would define as unbiased multidimensional deprivation incidence profile such that is unable to distinguish two population groups with no systematic differences in needs. However, as Fleurbaey and Schokkaert (2009) discussed, if this alternative condition is satisfied, it is possible that no difference in multidimensional deprivation incidence between two population groups will be observed because they have identical size of needs. This, despite these two populations still might have significant differences in deprivation. If such condition is attained the measure would depict these differently deprived populations under an equivalent multidimensional

Now, given that differences in deprivation originating strictly from differences in needs, cannot be straightforwardly differentiated from factual observed multidimensional deprivation incidence profiles because we do not know from the observed profile how much of the observed differences are due to differences in needs and how much of them are due to the measurement approach. We use a static microsimulation technique to generate a counterfactual deprivation profile in which the observed differences are strictly due to differences in needs. In such counterfactual scenario, no systematic difference in deprivation exists but only differences in needs.

Then, the evaluation of of our $m^{\beta,\theta}$ measures is approached as a 'controlled experiment' (a term used by Figari, Paulus, and Sutherland (2014) to describe microsimulation techniques) with the data to determine the ability of each measure to observe such counterfactual state of things and therefore to portrait an unbiased incidence profile.

The counterfactual scenario of no systematic difference in deprivation is created, in use of the 2013 Paraguayan Household Survey, by setting as invariant the characteristics of the household that describe differences in need and distributing deprivation completely at random across individuals and households. In other words, we fix the characteristics of the sample members (including whether or not they are members of applicable population subgroups) and then, for each j-dimension, we randomly allocate whether or not they are in deprivation. The random allocation is performed by sampling without replacement from the observed deprivation so that the total number of deprived people is the same in the counterfactual and factual samples.

The random distribution of deprivation emulates no systematic difference because is not related to any individual or household characteristics and thus is not a result of an underlying behaviour or characteristic. By building a (counterfactual) population in which there is no difference in deprivation resulting from these causes, we can determine whether a multidimensional incidence profile based on a particular $m^{\beta,\theta}$ measure is able to make an unbiased comparison.

Any multidimensional deprivation incidence profile satisfying the desirability condition, must exhibit no relation between multidimensional deprivation incidence and the size of household needs in this counterfactual scenario. Thus, we approach the evaluation of each profile in the counterfactual state of no systematic difference via a comparison of multidimensional deprivation incidence and the size of households needs. For this purpose, we use the linear regression $p_h = \rho + \delta N_h^0$, where p_h is the binary indicator of the presence or absence of multidimensional deprivation in the h-household, ρ is the intercept term, N_h^0 the count number of dimensions that the h-household needs, and δ is the regression coefficient of interest. This δ regression coefficient captures the difference in p_h-multidimensional deprivation incidence that can be attributed to the size of household needs. A profile that satisfies the desirability condition must reflect no difference in multidimensional deprivation incidence given by households'

deprivation incidence. For the purposes of this paper, this situation is considered ethically undesirable, so we deliberately focus on evaluating our measures only in terms of the selected desirability condition.

different needs.[7]

One could argue, nonetheless, that because of the randomness of the allocation of deprivation, a particular population subgroup might have a larger incidence of deprivation than another, simply as a result of this randomness. To overcome these possible random differences among population subgroups, the counterfactual scenario with no illegitimate difference in deprivation among households was simulated 1,000 times; each simulation or trial being independent from the other. The resulting collection of estimates approximates the distribution of the index over the counterfactual scenario's outcomes. The results that we describe below correspond to the distribution of these 1,000 independent simulations. For completeness and replicability purposes, Appendix A includes the implemented pseudo-code for these simulations.

4.4 Results

In this section, we describe the obtained results from this evaluation. Figure 2 on page 24 plots the summary of these results. The horizontal axis in the figure corresponds to the range of θ parameters used to calculate the $m^{\beta,\theta}$ measure. The first value of this range corresponds to $\theta = 0$ (no adjustment for the size of household needs), the adjustment by the size of household needs increases as θ increases. The last value on the right-hand side of the horizontal axis corresponds to $\theta = 1$. The vertical axis in the figure represents the estimated magnitude of the δ regression coefficient in p.p.

One estimated δ regression coefficient is obtained in each of the 1,000 simulations, and consequently, each δ coefficient measures the strength of the relationship between p_h-multidimensional deprivation incidence and the N_h^0-size of household needs in the counterfactual scenario of no systematic difference in deprivation. The 1,000 obtained δ coefficients describe the distribution of this relation in the (counterfactual) population in which there is no systematic difference in deprivation.

The mean of the 1,000 obtained regression coefficients is used as the measure of central tendency of the behaviour of δ. In Figure 2, each marker represents this central tendency measure of the δ regression coefficient obtained from using a particular $m^{\beta,\theta}$ measure. The shaded zone around the markers represents the range of variability of 95% of these 1,000 obtained estimates of δ. Any measure that properly accounts for differences in needs is, ideally, expected to have a

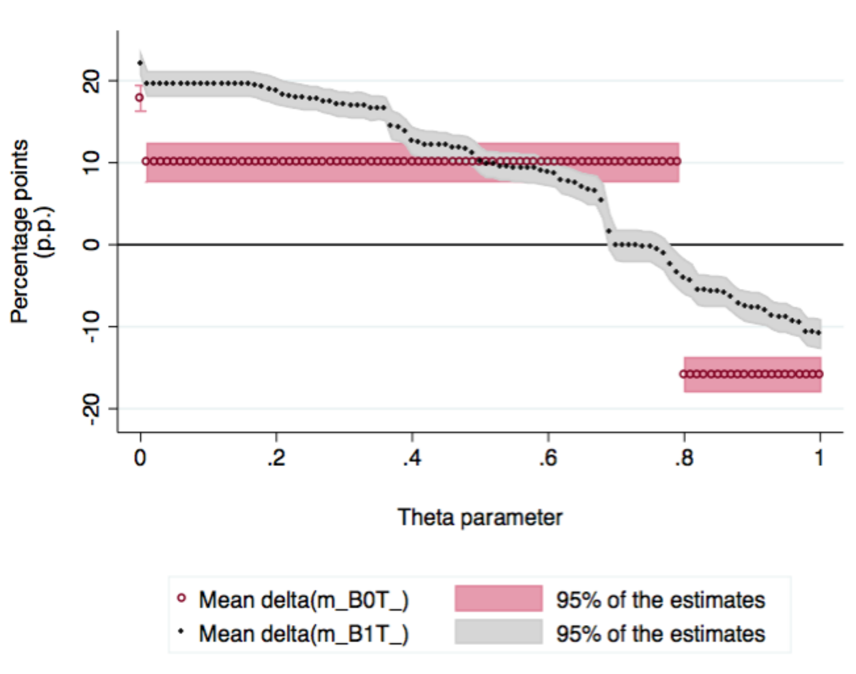
[7] Different approaches can nonetheless be used to measure the size of household needs, as for example household size. Still, the N_h^0-count number of dimensions that the h-household needs is our preferred measure of the size of household needs to be used for this evaluation because with such an approach the number of persons in the household and its composition is taken into account with respect to the dimensions captured by the multidimensional index. For a simple example consider households A and B, both consisting of two persons each. Household A, consisting of one adult person and one toddler. In the index example, this household may be scored as deprived in four out of the five considered dimensions. In contrast, household B, consisting of one adult and a 10-year-old child, may be scored as deprived in all five considered dimensions. In this case, household size does not capture the difference in possible deprivations that these two households of the same size have.

distribution with a mean of zero and a narrow spread (such as 95% of the values within that narrow interval).

When using the AF dimensions count-based approach to measurement to sort and identify households ($m^{0,0}$), we obtain across the 1,000 simulated scenarios a mean of the δ regression coefficient of 17.8 p.p., with a range of variability of 95% of its values between 16.3 and 19.4 p.p. This result indicates that comparing households on the basis of an $m^{0,0}$ metric does not permit an unbiased incidence profile. The simulation results of using this metric show a distribution of estimates far above the desirable zero mean, and their values are concentrated around this positive mean.

Similarly, the mean across the 1,000 simulations of the δ regression coefficient obtained when using the deprivations count-based approach to measurement ($m^{1,0}$), results in 21.9 p.p., with 95% of its values between 20.7 and 23.2 p.p.

Figure 2: Simulation results: distribution of the obtained δ regression coefficient in p.p. when using $m^{\beta,\theta}$ to sort and identify the most deprived households



Source: author's calculations based on 2013 Paraguayan Household Survey (PHS). Notes: Estimated population means based on a sample of 5,423 households. Results obtained by simulating 1,000 independent times a random allocation of deprivation across the observed households, keeping constant the demographic configuration of the households and the societal amount of deprivation in each indicator. Shaded areas denote 95% of the obtained δ estimates. The lower limit corresponds to the δ value at the 2.5 percentile and the upper limit to the δ value at the 97.5 percentile.

A positive δ regression coefficient observed across all 1,000 counterfactual scenarios when measuring the burden of multidimensional deprivation by any of these two metrics (the dimensions count-based approach and the deprivations count-based approach) indicates that both of these metrics produce multidimensional deprivation incidences p_h to be correlated with the size of the household needs. These results demonstrate that count-based measures cause any two households with different sizes of household needs to show different multidimensional deprivation incidence even if there is no systematic difference in deprivation between the two of them. Thus, these two metrics are shown to be ineffective capturing a state in which there are no systematic differences in deprivation between households but only differences in need.

On the other hand, sorting households using share-based approaches to measurement, as the $m^{0,1}$ or the $m^{1,1}$ metrics do, results as well in biased multidimensional deprivation incidence profiles. The distribution of the obtained δ regression coefficient in these two cases is concentrated below zero, the mean of both distributions have a value lower than -10 p.p and 95% of their values are concentrated around this negative mean. A negative mean across the simulations of the δ regression coefficient indicates that the metric used to sort and identify households does not effectively address differences in need. It produces a multidimensional deprivation incidence that decreases as the size of household needs increases.

For example, sorting households under the basis of the deprivations share-based approach to measurement ($m^{1,1}$) produces a distribution of the 1,000 obtained δ regression coefficients concentrated around 10.9 p.p. below zero, and the distribution of 95% of the estimates varies between -12.6 and -9.1 p.p. This means that, even when there is no systematic difference in deprivation between households, the use of an $m^{1,1}$ measure produces an additional dimension that the household exhibits as need to reduce the ability of this household to be classified as multidimensionally deprived by about 10.9 p.p.

Whereas count-based approaches cause a biased picture of household-based multidimensional deprivation profiles, larger and more heterogeneous households are more likely to be identified as the most deprived. Share-based approaches invert these results, also producing a biased picture of household-based multidimensional deprivation profiles. In the latter case, in contrast to count-based approaches, small and homogeneous households tend to be more likely to be identified as the most deprived, but only about half as often as in count-based approaches.

Nonetheless, sorting households in these counterfactual states based on any $m^{\beta,\theta}$ measure that uses $\beta = 1$ and a value θ between 0.69 and 0.77 satisfies the desirability condition. Any of these metrics produces a distribution of the obtained 1,000 δ regression coefficients between $p_h(m_h^{\beta,\theta})$ and N_h^0 with values very close to zero and a narrow spread of the distribution around this value. These results suggest that, for the particular case of the 2013 Paraguayan index example, these metrics enable to classify equivalently as either multidimensionally deprived or non-multidimensionally deprived households with no systematic difference in deprivation but only differences in needs among them.

One could argue however that the proposed linear regression approach to compare p_h and the size of household needs in the counterfactual scenario might be mistaken because such an approach could reveal no relation between p_h and the size of household needs, but still there could be an underlying relationship between them that is not necessarily linear. To analyse these possible alternative situations, the most straightforward recommended evaluation is to build independent simulations of the counterfactual state as is done in this paper, but rather than using a linear regression to measure the relationship between p_h and the size of household needs, using dummy variables of different household sizes to regress them against the p_h indicator obtained for the counterfactual scenario. The joint significance of the estimated relationship between the dummy variables and p_h can be tested as equal between each other and to zero. This is repeated for each independently simulated counterfactual scenario.

To illustrate this case, we built seven dummy variables of different household sizes and regressed p_h against six of them. This was performed for each of the 1,000 simulated counterfactual scenarios. The results for the particular case of the $m^{0,0}$ metric, show that in all 1,000 regressions the null hypothesis of equal relationship between the dummy variables and p_h is rejected. If, for example, the same approach is used to evaluate a profile based on an $m^{1,0.87}$ metric, in 80.8% of the 1,000 counterfactual scenarios, we cannot reject the null hypothesis of an equal estimated relationship among the six dummy variables for household size and p_h. These 1,000 consistent results confirm that measuring the burden of multidimensional deprivation with an $m^{0,0}$ metric does not permit households of different size with no systematic difference in deprivation to be classified as equivalently deprived. Consequently, we can assert the $m^{0,0}$ metric does not permit an unbiased multidimensional deprivation incidence profile in our 2013 Paraguayan application.

However, how do these results translate into factual observed profiles? To assess the implications of using the $m^{0,0}$ metric with respect to a metric demonstrating the ability to effectively capture differences in need, as for example in this Paraguayan context, the $m^{1,0.87}$ metric. Table 3 presents the multidimensional deprivation incidence obtained upon identifying the most deprived households under the basis of these two metrics and by household size. This comparison indicates that the use of the $m^{0,0}$ metric to sort and identify multidimensionally deprived households results in 13.2% of households consisting of one person, as well as 7.6% of households consisting of two persons, to be excluded from the multidimensionally deprived population because of their demographic profile. In addition, 11.0% of the households consisting of seven or more persons and 13.2% of the households consisting of six persons are identified as multidimensionally deprived because of their particular size and composition.

In summary, the results shown in this section indicate that neglecting differences in needs, and in particular the use of a dimensions count-based approach to measurement, yields biased household-based multidimensional deprivation incidence profiles. Other different combinations of β and θ to describe the burden of household multidimensional deprivation in the context of the 2013 Paraguayan application have proved to reveal unbiased multidimensional deprivation inci-

Table 3: Proportion of households identified as multidimensionally deprived (%)

Persons per household	$H(m^{0,0})$	[95% Conf.	Int.]	$H(m^{1,0.87})$	[95% Conf.	Int.]	Difference	
1	28.8	23.8	33.8	41.9	35.8	48.0	-13.2	***
2	35.0	31.0	39.0	42.6	38.5	46.6	-7.6	***
3	31.0	27.7	34.2	33.0	29.6	36.4	-2.1	
4	35.4	31.3	39.4	32.4	28.7	36.0	3.0	**
5	40.3	35.9	44.8	35.6	31.4	39.9	4.7	***
6	48.8	43.3	54.3	35.6	29.5	41.8	13.2	***
7 or more	58.7	53.0	64.5	47.7	42.3	53.2	11.0	***

Source: author's calculations based on 2013 PHS. Notes: Standard errors calculated using the Taylor linearised variance estimation and the stratified two-stage, clustered probabilistic sample design. ******* denotes a statistical significance level of 0.01. ****** denotes a statistical significance level of 0.05.

dence profiles. The degree to which we must account for these differences in need therefore stands out as relevant.

Still, these results correspond to the Paraguayan index example without applying any $\mathbf{w}$ dimensional weighting system and identifying the 40% most deprived households as the multidimensionally deprived population. The next section analyses the robustness of these 'baseline' obtained results under alternative considerations.

4.5 Alternative specifications

The first set of alternative specifications analysed in this section aims to build the index through a combination of indicators that 'balance' the applicable population subgroups within each dimension. This type of balancing procedure was proposed by Alkire (2015) as an alternative methodological approach to account for differences in need. It implies that each well-being dimension accounts all population subgroups with one applicable deprivation indicator. Note that, when introducing the five considered indicators for the Paraguayan illustration (Table 1 above), only the dwelling conditions dimension includes a set of indicators that balance the applicable populations subgroups. In contrast, the access to health services and education dimensions both include a set of indicators that together do not cover all population subgroups.

To illustrate the effect and implications of implementing balancing procedures of the type proposed by Alkire (2015), the first alternative specification analysed here consists of balancing the access to health services dimension. Specifically, the access to health services dimension can be considered as 'unbalanced' in the baseline configuration of the index, because the indicator of non-access to health services when needed applies exclusively to persons that were sick or had an accident during the 90 days before the interview; and such a dimension does not include any additional indicator for persons that were not sick or did not have an accident during the 90 days before the interview. Implementing a balancing procedure in this dimension implies either excluding the non-access to health services indicator from the index, or including an indicator applicable exclusively to the population that were not sick and had not had an accident during the 90 days before the interview. Here, the first approach is implemented because the latter would be conducive to including an indicator that is neither straightforwardly intuitive nor relevant for the purposes of policy.

The results of the δ relation coefficient between p_h and the size of household needs obtained using the Paraguayan index under this first alternative specification (i.e. excluding the non-access to health services when needed indicator) and across 1,000 counterfactual scenarios are shown in the second row of Table 4 (Specification A). In comparison to the baseline results, the magnitude of δ decreases while reducing the number of indicators applicable to specific population subgroups in the index. Whereas the baseline specification consists of aggregating five indicators, of which three describe differences in needs across demographic population subgroups, this alternative Specification A includes only two out of these three indicators. This reduction in indicators, in the case of the dimensions count-based approach to measurement, results in a mean δ 7.6 p.p. smaller than that observed in the baseline specification.

It is worth noting that this first alternative specification of the index balances the access to health services and the dwelling conditions dimensions, but the education dimension remains unbalanced. To balance this remaining dimension, an additional indicator applicable exclusively to children under five years of age is included. The additional deprivation indicator included corresponds to whether or not a 0âĂŞ4 years old child has been registered into the national identification system. The results of this specification are labelled in Table 4 as Specification B.

The results of Specification B show a larger mean δ relation than the obtained by Specification A. This confirms that the strength of the relation between multidimensional deprivation incidence and the size of household needs increases/decreases along the number of indicators that depict differences in need across population subgroups does.

Interestingly, the results of Specification B also indicate that despite population subgroups per each dimension being completely balanced across indicators, the obtained distribution of the δ-relation between multidimensional deprivation incidence and the size of household needs is located far above zero with its values concentrated around 16.9 p.p. This result proves that a dimensions count-based approach to measurement ($m^{0,0}$) is unable to classify equivalently any two households with no systematic difference in deprivation but only different needs. This is even though balancing procedures across dimensions and indicators have taken place.

Still, one might argue that a weighting structure to avoid giving more importance to some dimensions over others could enhance the outcome of this type of balancing procedures. Hence, the third alternative specification tested here imposes over Specification B a nested weighting structure, which means each dimension and each indicator within each dimension has an equal relative importance in the index. The results of this fourth specification are shown in the table 4 as Specification C. We observe that this weighting structure rather worsens the results and increases in about 70% the size of the δ-relation with regard to the unweighted specification (Specification B).[8] This result is observed because the implemented weighting system of Specification C reduces the importance in the index of the indicators that apply to any person (health insurance noncoverage and substandard dwelling conditions) and increases the importance of indicators that capture differences in needs across population subgroups.

Another possible course of action to implement a balancing procedure could consist of applying a set of weights that vary across population subgroups that exhibit different sets of needs. In the Paraguayan example, this procedure implies imposing over the balanced specification of the index (Specification B) a set of weights such that the sum of relative importances across indicators is one in each population group that is accounted through a different set of indicators. Note that here weights are applied over individuals' deprivations rather than over household's dimensions. The results of this D-alternative balancing specification show a δ relation equivalent to the obtained using the unweighted balanced specification of the index (Specification B).

Now, in terms of different shares of the population to identify the multidimensionally deprived population, while the baseline results are based on identifying the first 40% most deprived households, here as alternative specifications 20% and 30% of the total population of households are identified as the multidimensionally deprived. Results are shown in the last two rows of Table 4. As expected, the size of the δ relation coefficient is sensitive to the share of population identified as multidimensionally deprived, it decreases as the share of identified multidimensionally deprived population decreases.

In summary, these alternative specifications, consistently with Section 4.4's findings, demonstrate that measuring the burden of multidimensional deprivation without accounting for differences in need, as the dimensions count-based approach to measurement does, produces a biased multidimensional deprivation incidence profile. It captures not only relevant differences in deprivation but also unaddressed differences in needs. Though a deprivations share-based approach addresses differences in needs, this approach to measurement overshoots the results. The obtained δ estimates that use this relative approach to measurement result concentrated not around the desired zero mean, but around negative values. It conduces multidimensional deprivation incidence to decrease systematically as the size of the household needs increases. Still, these nega-

[8] An unweighted specification across five indicators implies in practice that each indicator has one fifth of relative importance in the whole index.

Table 4: Simulation results: distribution of the obtained δ regression coefficient in p.p. when using $m^{\beta,\theta}$ to sort and identify the most deprived households

	Specification	Dimensions count-based ($m^{0,0}$)			Deprivations share-based ($m^{1,1}$)		
		Mean δ	Ll	Ul	Mean δ	Ll	Ul
		(1)	(2)	(3)	(4)	(5)	(6)
	Baseline						
	Two unbalanced dimensions & 40% of share	17.9	16.5	19.5	-10.9	-12.7	-9.2
	Alternative specifications						
A	One unbalanced dimension	10.3	7.8	12.8	-16.9	-19.5	-14.2
B	Balanced specification: All dimensions are balanced	16.9	15.2	18.7	-9.6	-11.3	-7.6
C	Weighted balanced specification	29.0	25.7	32.2	-14.4	-17.9	-10.7
D	Using an individual weighting system	17.0	15.2	18.6	-9.5	-11.3	-7.5
E	20% of share	12.0	10.9	13.3	-8.0	-9.6	-6.5
F	30% of share	16.3	14.9	17.6	-8.9	-10.5	-7.3

Source: author's calculations based on 2013 PHS. Notes: Estimated population means based on a sample of 5,423 households. Results obtained by simulating 1,000 independent times a random allocation of deprivation across the observed households, keeping constant the demographic configuration of the households and the societal amount of deprivation in each indicator. Columns (1) and (4) correspond to the mean δ regression coefficient across these 1,000 simulations. Columns (2) and (5) correspond to the δ value at the 2.5 percentile of the distribution of δ coefficients across the 1,000 performed simulations. Columns (3) and (6) correspond to the δ value at the 97.5 percentile.

tive mean δ values are about half the size of those obtained by the dimensions count-based metric.

In next section we briefly discuss the case when multidimensional deprivation is evaluated at the individual level; the methodology presented in Section 3.2 is called 'the individual-based method'.

4.6 The individual-based method

This section illustrates the empirical behaviour of the multidimensional deprivation measurement methodology proposed in this paper when using individual as the unit of multidimensional deprivation analysis and evaluates its proposed measures. For this purpose, the same 2013 PHS indicators used for analysis in previous sections are used here. However, as described when outlining the methodology for the individual-based method (Section 3.2), household-based aggregates are not pursued here. In contrast, in the individual-based method, each individual is considered its own household and the burden of multidimensional deprivation is measured by an m^θ metric.

Table 5 presents for each 2013 PHS considered deprivation indicator the number of observed persons in its applicable population and the proportion of deprived persons within it. These five deprivation indicators are subsequently combined to depict the burden that multidimensional deprivation places on each individual. As a result, we obtain the m^θ index that takes values according to the used θ parameter of responsiveness of deprivation with regard to the level of needs.

To evaluate measures, we use the same desirability condition used in the afore-presented sections and outlined on page 21. As such, in a counterfactual scenario with no systematic difference in deprivation among individuals but only differences in needs across them, we analyse the relationship between the multidimensional deprivation incidence (p_i) and the size of individual needs. Similarly to the household-based case, we approach this analysis via the linear regression $p_i = \rho + \delta N_i$, where N_i represents the size of individuals multidimensional needs and is measured by the number of achievements, i.e., indicators,

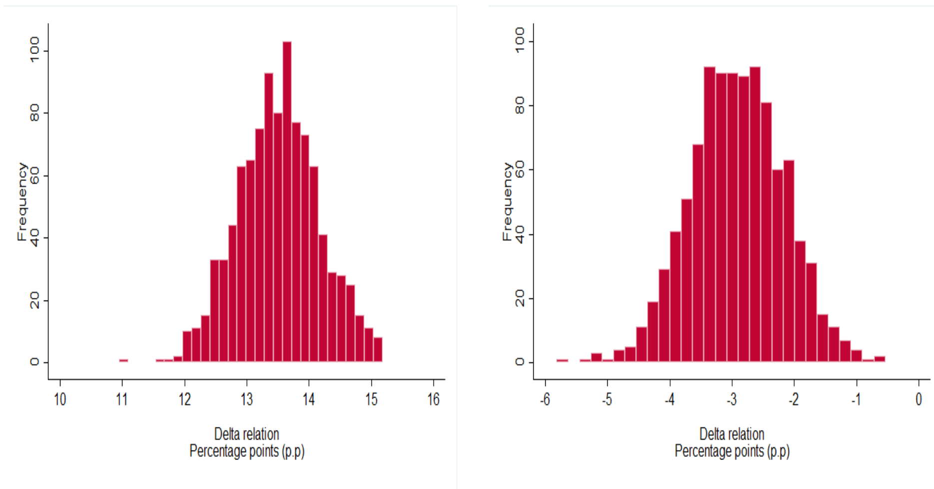
Table 5: Observed individual dimensional deprivation

Deprivation indicator	Applicable population where the indicator is relevant	Number of observed persons in the applicable population	Proportion (%) of deprived persons in the applicable population
Health insurance non-coverage	Any person	20,909	71.9
No access to health services	Any person that was sick or had an accident during the 90 days previous to the interview	7,199	23.8
Non-school attendance	5 - 17 years old population	5,706	6.7
Low educational achievement	18 years old population and over	13,406	47.9
Sub-standard housing	Any person	20,909	24.6

Source: author's calculations based on 2013 PHS.

Figure 3: Simulation results: distribution of the obtained δ regression coefficient in percentage points (p.p.) using two different m^θ measures

(a) m^0 (b) m^1



Source: author's calculations based on 2013 PHS. Notes: Estimated population means based on a sample of 20,909 individuals. Results obtained by simulating 1,000 independent times a random allocation of deprivation across the observed individuals, keeping constant the demographic configuration of the population and the societal amount of deprivation in each indicator.

the individual exhibits in need.

Figure 3 plots the evaluation results of m^0 and m^1. The distribution of the obtained δ estimated coefficient, when using m^0 to rank individuals, register positive values across all the 1,000 simulated scenarios and ranges between 11.0 and 15.2 p.p. In these counterfactual scenarios of randomly allocated deprivation, one additional dimension increases the multidimensional deprivation incidence by an average of 13.5 p.p. These results indicate that the m^0 metric leads to a biased multidimensional deprivation incidence profile.

Similarly, a multidimensional profile based on a share-based measure does not satisfy the desirability condition. We observed from Figure 3.b that the estimates of the δ regression coefficient across the 1,000 simulations range between -5.8 and -0.5 p.p., and the mean of these estimates is concentrated at 2.9 p.p. below zero.

The evaluation results of these two metrics (m^0 and m^1) were obtained upon identifying as multidimensionally deprived the 40% most deprived population. Other alternative population shares were also used to identify the multidimensionally deprived population (30% and 20%), and the results proved robust under these other two population shares.

We now proceed to evaluate the proposed family of societal measures of this paper in terms of the characteristics that make it suitable for the purpose of multidimensional deprivation measurement.

5 Properties

Not every societal multidimensional deprivation measure is meant to satisfy all desirable properties. The characterization of the family of measures of this paper, critically draws and extends the properties proposed by Alkire and Foster (2011) for their family of multidimensional measures of poverty. As such, in this section we characterize the members of the proposed family of measures of this paper in terms of the properties that make them non-sensitive to non-relevant aspects of the distribution and also in terms of the properties that reflect the measures' proper orientation.

5.1 Non-relevant sensitivities

The poverty measurement literature and, in particular, the multidimensional literature characterises some families of societal measures under a *symmetry* or *anonymity* property. According to Alkire and Foster (2011), the symmetry property that their family of measures uses, ensures that societal metrics are not being constructed under the basis of greater emphasis on some population subgroups over others. This property is also used by multidimensional measures, such as the ones proposed by Tsui (2002), Bourguignon and Chakravarty (2003), and Seth (2010), among others. In particular, Bourguignon and Chakravarty (2003) defined their measures as symmetric since any person's characteristics, other than the multiple well-being dimensions considered for the measure, are set as not relevant in the measurement process of their measures. Similarly, Seth (2010) imposes the identities of the individuals as ethically not significant for the measurement process and therefore individuals are considered anonymous within society. We henceforth refer to this measurement property as *anonymity.*

However, in the presence of differences in needs across population subgroups, the individual or household heterogeneity conflicts with this property because individuals can not longer be assumed as anonymous and the characteristics that define their needs must be taken into account. The one-dimensional welfare measurement literature has tackled this conflict by either measuring each personâĂŹs well-being through a common metric that incorporates the information on the heterogeneity in needs and then aggregating across persons using the anonymity property or, alternatively, by dropping the anonymity property and accounting for the heterogeneity with, for instance, a weighting system that reflects those heterogeneous needs (Coulter, Cowell, and Jenkins, 1992b). In line with the former approach, the methodological proposal of this paper tackles differences in needs by incorporating the information of this heterogeneity in the measures and imposes the anonymity property over the resulting societal metrics.

Therefore, our societal measures are meant to be non-sensitive to permutations of the units where the identification of the multidimensionally deprived population occurs. In the individual-based method, this means that societal measures are non-sensitive to rearrangements of individuals across the population. In the household-based scenario, societal measures are then meant to be non-sensitive to permutations of households within society. This characteristic is termed, for the purposes of this paper, as *household anonymity.*

Still, household anonymity does not imply measures to be non-sensitive to permutations of individuals across households. Permutations of individuals across households resemble either demographic changes or transfers across units, to which the proposed household-based measures of this paper are sensitive. We further elaborate on these sensitivities when discussing the dominance and scale invariance properties proposed for our metrics.

Consequently, we impose over our societal metrics a property of non-sensitivity to permutations of individuals within households, which we call *Within household anonymity.* The usage of within household anonymity implies that the intra-household bargaining power and differences in allocation of resources at the household level are not taken into account when measuring the burden that deprivation places over each individual. Our approach to measurement assumes that the burden of multidimensional deprivation is equally experienced across household members. On the contrary, one branch of the one-dimensional welfare measurement literature shows that despite individuals behave under collective instances in the household, they have significant differences in preferences and in their achieved welfare level. Examples of this literature are the studies of Chiappori (1992), Browning, Chiappori, and Lewbel (2013), and Mazzocco (2005). This measurement issue requires further research to be addressed in the multidimensional deprivation context, which we mark as a future research endeavour.

We begin the formal presentation of the household anonymity, the within household anonymity property and the subsequent properties as follows:

Let a population of $I \geq 2$ individuals belonging to $R \geq 1$ households, and an $I \times J$ matrix $\mathbf{G} = [g_{ij}(s_j)]$ of individual dimensional deprivation, where $g_{ij}(s_j)$ is the binary indicator of presence or absence of deprivation for the i individual in the j dimension as defined by (4). The h-household is q_h sized and each individual's household membership is known.

1. **Household anonymity.** Let the matrix $\mathbf{G}'$ of dimensional deprivation that household members experience be defined as a variant of the $\mathbf{G}$ matrix such that each row of $\mathbf{G}'$ corresponds to a household and each column to the binary indicator of presence or absence of deprivation of the i individual that belongs to the h household at the j dimension. If we obtain the matrix $\mathbf{G}''$ from a permutation such that $\mathbf{G}'' = \pi\mathbf{G}'$ where, π is a square $R \times R$ matrix that has one entry 1 in each row and each column, and 0 in the remaining positions, then $MD^{\beta,\theta}(\mathbf{G}) = MD^{\beta,\theta}(\mathbf{G}'')$.

2. **Within household anonymity.** Let $\mathbf{G_h}$ be a $q_h \times J$ sub-matrix of $\mathbf{G}$ that contains the individual dimensional deprivation of the household members that belong to the h household. If $\mathbf{G_h''}$ is obtained from a permutation such that $\mathbf{G_h''} = \pi_h\mathbf{G_h}$ where, π_h is a square $q_h \times q_h$ matrix that has one entry 1 in each row and each column, and 0 in the remaining positions, then $MD^{\beta,\theta}(\mathbf{G_h}) = MD^{\beta,\theta}(\mathbf{G_h'})$.

Further, the income or expenditure poverty measurement literature has traditionally used a *population replication invariance* to ensure societies of different population size can be compared in terms of their incidence, depth and severity of poverty. Along to this property a *normalization* property is traditionally used to express measures in relation to the poverty line. In particular, this is the approach used by Foster et al. (1984) and described in detail by Foster (2006). For the case of the family of measures proposed in this paper a population replication invariance is imposed to allow comparisons of any two societies with different amount of multidimensional needs and a normalization property to ensures measures to be expressed in relation to the size of the household needs. These two properties are formalized as follows:

3. **Population replication invariance.** For a population of R households, if we replicate $t \geq 2$ times these R households, the society level multidimensional measure $MD^{\beta,\theta}$ will remain unchanged.

4. **Normalization.** If all household members in society have all their applicable dimensions in deprivation, which is $g_{ij}(s_j) = 1$ for any i individual in a j dimension satisfying $s_{ij} = 1$, then $MD^{\beta,\theta} = 1$. Conversely, if for all household members in society have all their applicable dimensions absent of deprivation, which is $g_{ij}(s_j) = 0$ for any i individual in a j dimension satisfying $s_{ij} = 1$, then $MD^{\beta,\theta} = 0$.

Despite population replication invariance and normalization have been traditionally used in the poverty measurement literature, these properties can be controversial. As pointed out by Duclos and Tiberti (2016), if we are interested in comparing the deprivation's breadth across two societies these two properties are not necessarily desirable. The family of measures proposed in this paper overcomes this drawback by incorporating not only societal measures that use a relative approach to measurement, but also other members that use an absolute perspective. These members of our societal measures allow ranking societies in terms of their absolute size of deprivation and correspond to any $MD^{\beta,\theta}$ such that $\theta = 0$.

As such, exclusively H and $MD^{\beta,\theta}$ measures with $\theta = 1$, correspond to a relative approach to measurement and are invariant to the size of the population to be compared. These measures are the only ones that attain the normalization property and therefore have a range of variability within the interval $[0,1]$. While H expresses the incidence of multidimensional deprivation in relation to the size of the population of households, $MD^{\beta,1}$ expresses multidimensional deprivation in relation to the size of household needs. We now state the following result regarding these properties.

Lemma 5.1. *For any given k-multidimensional deprivation threshold and any $\mathbf{w}$ vector of weights, the multidimensional deprivation measure $MD^{\beta,\theta}$ satisfies household anonymity, within household anonymity and population replication invariance for any $\beta \in \{0,1\}$ and $\theta \in [0,1]$ and normalization if and only if $\theta = 1$.*

Proof. See Appendix B. $\square$

Other non-relevant sensitivities of our proposed $MD^{\beta,\theta}$ societal measures are the following types of increments in achievement levels: i) increments in the j achievement level among individuals that belong to a non-multidimensionally deprived household; ii) increments in the j achievement level among j non-deprived individuals; and iii) increments in the j achievement level among individuals that do not belong to the j applicable population subgroup. We continue further elaborating on this.

The non-sensitivity of the measures to increments in achievement levels among individuals that belong to non-multidimensionally deprived households, in the context of this paper, is termed *multidimensional deprivation focus.* In comparison to the AF method, this characteristic is analogous to the poverty focus property proposed by the AF method and is the following:

5. **Multidimensional deprivation focus.** Let a population of $R \geq 1$ households, made of I individuals and an $I \times J$ matrix $\mathbf{A} = [a_{ij}]$ of individual achievements. If a'_{ij} is obtained by an increment of a constant $\gamma > 0$ such that $a'_{ij} = a_{ij} + \gamma$ for any person i that belongs to a non-multidimensionally deprived household while the others remain the same, i.e., $a'_{ij} = a_{ij}$, then $MD^{\beta,\theta}(\mathbf{A}) = MD^{\beta,\theta}(\mathbf{A}')$.

It is worth noting, however, that in a household-based scenario, this multidimensional deprivation focus property enforces societal measures to be non-sensitive to increments in achievements of both deprived and non-deprived individuals that belong to non-multidimensionally deprived households. Identifying the multidimensionally deprived population at the household level is based on considering the household as a single unit. As such, it prevents observing multidimensionally deprived and non-multidimensionally deprived individuals within the same household. It produces measures to be non-sensitive to improvements or declines in achievement levels of deprived individuals that might have a large number of dimensions in deprivation but that do not belong to multidimensionally deprived households. This is in fact the case of any societal measures based on household-based metrics, either $H(m^{0,0})$ and M_0 from the AF method or the proposed $H(m^{\beta,\theta})$ and $MD^{\beta,\theta}$ measures of this paper.

On the other hand, as discussed when introducing the proposed methodology of this paper, every achievement is not necessarily relevant to be measured across an i person. Then, the $MD^{\beta,\theta}$ proposed measures uncover this consideration. This means that the $MD^{\beta,\theta}$ family of measures considers non-relevant achievement increments, not only among j non-deprived individuals but also among individuals that do not belong to the j applicable population. In the context of this paper, this property is called *applicable deprivation focus* and is formalized as follows:

6. **Applicable deprivation focus.** Let a population of $R \geq 1$ households, made of I individuals and any $I \times J$ matrix $\mathbf{A} = [a_{ij}]$ of individual achievements. If a'_{ij} is obtained by an increment of a constant $\gamma > 0$ such that $a'_{ij} = a_{ij} + \gamma$ for any person i such that $a_{ij} \geq z_j$ or any person i in a

j dimension satisfying $s_{ij} = 1$, while the others remain $a'_{ij} = a_{ij}$; then, $MD^{\beta,\theta}(\mathbf{A}) = MD^{\beta,\theta}(\mathbf{A}')$.

We now present and prove the following Lemma regarding these two properties.

Lemma 5.2. *For any given k-multidimensional deprivation threshold and any $\mathbf{w}$ vector of weights, the multidimensional deprivation measure $MD^{\beta,\theta}$ satisfies multidimensional deprivation focus and applicable deprivation focus for any $\beta \in \{0,1\}$ and $\theta \in [0,1]$.*

Proof. See Appendix B. □

We now discuss the set of properties that describe the orientation and desirable sensitivities of our measures, which are termed by Foster (2006) as *dominance properties*. This set of properties reflect the ability of our metrics to describe improvements or declines among the multidimensionally deprived population.

5.2 Dominance properties

We begin this characterization defining three relevant types of improvements in achievement levels: i) *applicable achievement increment*, ii) *deprivation reduction among the multidimensionally deprived*, and iii) *dimensional deprivation reduction among the multidimensionally deprived*.

An *applicable achievement increment* occurs whenever the i individual that belongs to the s_j applicable population subgroup increases its a_{ij} achievement level by a constant $\gamma > 0$. This means that the a'_{ij} achievement for the i household-member and the j dimension is obtained by an increment of a constant $\gamma > 0$, such that $a'_{ij} = a_{ij} + \gamma$ for any person i in a j dimension satisfying $s_{ij} = 1$.

Now, assume that this i individual is deprived in the j dimension and belongs to a multidimensionally deprived household. Then, a *deprivation reduction among the multidimensionally deprived* occurs whenever this i individual increases his/her welfare in the j achievement, and this improvement changes his/her status in the j dimension from deprived to non-deprived. Hence, in addition to be an applicable achievement increment, a deprivation reduction amongst to the multidimensionally deprived, makes this individual no longer j-deprived due to this welfare improvement. This means that the a'_{ij} achievement for the i individual in the j dimension, obtained by an increment of a constant $\gamma > 0$ such that $a'_{ij} = a_{ij} + \gamma$, for any person i in a j dimension satisfying $s_{ij} = 1$, $a_{ij} < z_j$, $i \in h$, given that $p_h = 1$, is a deprivation reduction among the multidimensionally deprived whenever $a'_{ij} \geq z_j > a_{ij}$.

Nonetheless, the household to which this i individual belongs might be still having any other household member j-deprived, therefore continuing to be deprived in such dimension. Then, a *dimensional deprivation reduction among* *the multidimensionally deprived* is an improvement such that it involves an applicable achievement increment that produces a deprivation reduction among the multidimensionally deprived and also a change in the household status from deprived to non-deprived in such a dimension. This means that the a'_{ij} achievement for the i individual and the j dimension, obtained by an increment of a constant $\gamma > 0$ such that $a'_{ij} = a_{ij} + \gamma$ for any person i in a j dimension satisfying $s_{ij} = 1$, $a_{ij} < z_j$, $i \in h$, given that $p_h = 1$, produces $a'_{ij} \geq z_j > a_{ij}$ and $d_{hj}^{\beta'} = 0$, where the d_{hj}^{β}-dimensional deprivation indicator for the h household and the j dimension is defined as per in (5) and before this achievement increment takes a $d_{hj}^{\beta} > 0$ value. After the achievement increment this deprivation indicator corresponds to $d_{hj}^{\beta'} = 0$.

Having introduced these three different types of increases in welfare as relevant, we propose the following three properties to characterise our $MD^{\beta,\theta}$ family of measures:

7. **Weak achievement monotonicity.** Let a population of $R \geq 1$ households, made of I individuals and an $I \times J$ matrix $\mathbf{A} = [a_{ij}]$ of individual achievements. If a'_{ij} is obtained from a_{ij} by an applicable achievement increment, then $MD^{\beta,\theta}(\mathbf{A}) \geq MD^{\beta,\theta}(\mathbf{A}')$.

8. **Deprivation monotonicity.** If a'_{ij} is obtained from a_{ij} by a deprivation reduction among the multidimensionally deprived, then $MD^{\beta,\theta}(\mathbf{A}) > MD^{\beta,\theta}(\mathbf{A}')$.

9. **Dimensional deprivation monotonicity.** If a'_{ij} is obtained from a_{ij} by a dimensional deprivation reduction amongst the multidimensionally deprived, then $MD^{1,\theta}(\mathbf{A}) > MD^{1,\theta}(\mathbf{A}')$.

We proceed to show that the above properties are valid.

Lemma 5.3. *For any given k-multidimensional deprivation threshold and any $\mathbf{w}$ vector of weights, the multidimensional deprivation measure $MD^{\beta,\theta}$ satisfies weak achievement monotonicity and dimensional deprivation monotonicity for any $\beta \in \{0,1\}$ and $\theta \in [0,1]$, and deprivation monotonicity if and only if $\beta = 1$.*

Proof. See Appendix B. □

Not every measure satisfies the three proposed dominance properties, each combination of β and θ parameters enforces different properties. As such, the measurement of household multidimensional deprivation using our $MD^{\beta,\theta}$ family of measures and setting $\beta = 1$, which means counting deprivations in the household rather than dimensions in deprivation, have proved to ensure the desirable orientation.

In terms of an individual-based method and in comparison to the AF set of properties, the proposed weak achievement monotonicity and dimensional deprivation monotonicity properties of this paper result equivalently to the AF's proposed *weak monotonicity* and *dimensional monotonicity* properties, respectively. However, the AF metrics do not satisfy deprivation monotonicity, which make their measures non-sensitive to increments in deprivation among the multidimensionally deprived population. Feature not desirable for the purposes of poverty measurement.

Another important dominance characteristic for multidimensional deprivation measurement is the ability of the measure to fall unambiguously under any applicable achievement increment, even if such achievement increment does not remove deprivation. The AF method termed this property as *monotonicity.* In the case of the AF method, any M_α with $\alpha > 0$ is meant to satisfy monotonicity. However, M_α with $\alpha > 0$ are metrics not commonly used in the applied literature because they require that all considered achievement indicators being cardinal. Given the ordinal nature of the majority of policy indicators, the AF's monotonicity property is therefore hardly exhibited. In terms of the measures proposed in this paper, any measure satisfying monotonicity would produce a decrease in the societal value of $MD^{\beta,\theta}$ due to an applicable achievement increment. But, the ordinal nature of policy indicators conduces to express them in terms of binary data, which disables the ability to document this type of welfare improvement.

5.3 Decomposability properties

Any metric that can be expressed as a weighted average of subgroup estimates, where weights are population subgroup shares can be defined as decomposable. Articles such as Foster et al. (1984), Tsui (1999), and Alkire and Foster (2011) refer to this property as *decomposability*, and Bourguignon and Chakravarty (2003) refers to it as *subgroup decomposability.* In the context of our proposed methodology, decomposability is enabled across groups of units where the identification of the multidimensionally deprived population occurs. This means that measures that use individuals as the unit of analysis can be decomposed by subgroup of individuals and those that use household as the unit of analysis can be decomposed by subgroup of households. We name this property as *decomposability by household subgroups* and formalize it as follows:

10. Decomposability by household subgroups. Let a population of $I \geq 2$ individuals belonging to $R \geq 2$ households and a $I \times J$ matrix $\mathbf{G}$ of individual dimensional deprivation. For a given positive integer L let $G_1, G_2, \ldots, G_L$ be submatrices of $\mathbf{G}$ and further let $\lambda = 1, \ldots, L$. Each G_λ represents the $I_\lambda \times J$ matrix of dimensional deprivation for a subpopulation made of I_λ individuals that belong to R_λ households, with $\sum_{\lambda=1}^{L} I_\lambda = I$. Then, it holds that

$$MD^{\beta,\theta}(\mathbf{G}) = \sum_{\lambda=1}^{L} \frac{I_\lambda}{I} MD^{\beta,\theta}(G_\lambda).$$

The second worth analysing decomposability feature refers to the ability of societal metrics to be expressed as a weighted average of the contribution of each dimension into the overall multidimensional deprivation estimate. In the context of our proposed measures this property is formalized below

11. **Dimensional decomposability.** Let a population of $I \geq 2$ individuals belonging to $R \geq 2$ households and a $I \times J$ matrix $\mathbf{G}$ of individual dimensional deprivation, and let $G_1, G_2, \ldots, G_J$ submatrices of $\mathbf{G}$. Each G_j is the $I \times 1$ vector of j dimensional deprivation for the population of I individuals. Then for $\beta \in \{0,1\}$ and $\theta = 0$.

$$MD^{\beta,\theta}(\mathbf{G}) = \sum_{j=1}^{J} \frac{1}{I} MD^{\beta,\theta}(G_j).$$

Lemma 5.4 shows that our $MD^{\beta,0}$ measures satisfy the decomposability of household subgroups, which means they can be consistently decomposed into population subgroups and contributions. Also it shows that our $MD^{\beta,0}$ measures can be also decomposed to determine the contribution of each dimension into the over all multidimensional deprivation measurement.

Still, we remark that an identification of the multidimensionally deprived at the household level produces societal measures not being decomposable by population subgroups of individuals disregarding the household where they belong (i.e. ranges of age, gender, or ethnicity), which is the case of current household-based applications of multidimensional deprivation measurement as well.

Lemma 5.4. *For any given k-multidimensional deprivation threshold and any $\mathbf{w}$ vector of weights, the multidimensional deprivation measure $MD^{\beta,\theta}$ satisfies decomposability by household subgroups and dimensional decomposability for $\theta = 0$.*

Proof. See Appendix B. □

The imposition of the 11 afore-discussed properties leads to the following result that represents the characterization of our proposed family of measures as suitable for the purposes of multidimensional deprivation measurement. The following Theorem can be easily simplified for the individual-based method and follows from Lemmas 5.1, 5.2, 5.3 and 5.4.

Theorem 5.5. *For any given k-multidimensional deprivation threshold and any $\mathbf{w}$ vector of weights, the multidimensional deprivation measure $MD^{\beta,\theta}$ satisfies household anonymity, within household anonymity population replication invariance, decomposability by household subgroups, multidimensional deprivation focus, applicable deprivation focus, weak achievement monotonicity and dimensional deprivation monotonicity for any $\beta \in \{0,1\}$ and $\theta \in [0,1]$; normalization if and only if $\theta = 1$; dimensional decomposability and deprivation monotonicity if $\beta = \theta = 1$ or if $\beta = \{0,1\}$ and $\theta = 0$.*

The next subsection presents the properties that we impose over the measure of the burden of multidimensional deprivation, which enable policy makers to rank households from most to least deprived in terms of the depth of deprivation that any differently sized and composed household may exhibit.

5.4 Ranking households

When it comes to rank households and determine how much deprived the worst off household is with regard to the least deprived, the ability of the measures to meaningfully commensurate the size of deprivation becomes crucial, as well as assuring a well behaved distribution of the multiple deprivations. In this section we describe the three properties that we impose for this purpose: *cardinality*, *absolute translation invariance* and *equiproportional translation invariance.*

Continuity in the multidimensional measurement literature has been used, for instance, by Bourguignon and Chakravarty (2003) to characterise welfare multidimensional measures. It ensures a well-behaved functional form that would produce no abrupt jumps given small changes in achievements. The proposed approach of this paper may have more than one possible source of discontinuity because is based on binary data that typifies whether or not the person is deprived each achievement and uses a second cut-off point to identify the multidimensionally deprived population.

However, since most of the indicators in use for public policy purposes are $\{0, 1\}$ indicators of presence or absence of deprivation, the lack of continuity in the proposed metrics of this paper is seen as an actual depiction of the policy context. To the contrary, we exploit the ordinal nature of the indicators and use either count-based, share-based, or a mixture of both approaches to lead to a cardinal approach to express the $m^{\beta,\theta}$-burden of multidimensional deprivation.

The cardinal nature of our proposed $m^{\beta,\theta}$ metrics allows comparison of the size of the burden of multidimensional deprivation across any two given households, which for policy purposes constitutes a technology that allow not only ranking households from most deprived to least deprived but also to measure their difference in deprivation in terms of size as follows.

1. **Cardinality.** For any given two households A and B evaluated across J dimensions, A is said to be t times as deprived as B if and only if $m_A^{\beta,\theta} = t m_B^{\beta,\theta}$, for a positive number t.

On the other hand, to describe the distribution of multidimensional deprivation, we specifically draw from the properties proposed by Tsui (2002) and Kolm (1976) for measures of absolute and relative inequality and propose two types of non-relevant changes in the distribution of multidimensional deprivation: i) An addition in the same amount of deprivation to all households in society and ii) an equiproportional variation in multidimensional deprivation among all households in society. The non-sensitivity of the measures to these two non-relevant changes in the distribution are respectively termed as *absolute translation invariance* and *Equiproportional translation invariance*, and formalized as follows:

2. **Absolute translation invariance.** Let a population of $R \geq 2$ households evaluated across J dimensions through an $m_h^{\beta,0}$ measure of burden of multidimensional deprivation, and a given constant ζ share of the population identified as multidimensionally deprived. If $(m^{\beta,0})'$ is obtained by an increment of a constant $\gamma > 0$, such that $(m^{\beta,0})' = m_h^{\beta,0} + \gamma$ for all R

households, then, the proportion of households identified as multidimensionally deprived and the ranking of households using $(m^{\beta,0})'$ and $m_h^{\beta,0}$ remains invariant.

3. **Equiproportional translation invariance.** Let a population of $R \geq 2$ households evaluated across J dimensions through an $m_h^{\beta,\theta}$-burden of multidimensional deprivation, and any given constant ζ share of the population identified as multidimensionally deprived. If $(m^{\beta,\theta})'$ is obtained by an equiproportional variation in multidimensional deprivation by a constant $\gamma > 0$, such that $(m^{\beta,\theta})' = m_h^{\beta,\theta}\gamma$ for all R households, then, the proportion of households identified as multidimensionally deprived and the ranking of households using $m_h^{\beta,\theta}$ and $(m^{\beta,\theta})'$ remains invariant.

We show now that these properties hold regarding the measure $m^{\beta,\theta}$ of multidimensional deprivation.

Proposition 5.6. *For any given* $\mathbf{w}$ *vector of weights, the* $m^{\beta,\theta}$ *measure of multidimensional deprivation satisfies cardinality for any* $\beta \in \{0, 1\}$ *and* $\theta \in [0, 1]$*; absolute translation invariance if and only if* $\theta = 0$*; and equiproportional translation invariance if and only if* $\theta \in (0, 1]$.

Proof. See Appendix B. □

6 Context-specific definitions

Multidimensional deprivation measurement embeds, as does any poverty measurement process, several assumptions and normative definitions that vary from context to context. In fact, Sen (1979) indicates the following as a good practice for the general poverty measurement exercise: 'There is very little alternative to accepting the element of arbitrariness in the description of poverty, and making that element as explicit as possible.' (Sen, 1979, p.288). In light of this, in this section we discuss the most relevant context-specific definitions embedded in the proposed multidimensional deprivation family of indices.

Unit of multidimensional deprivation analysis. The first normative selection required when measuring multidimensional deprivation is the unit of analysis where multidimensional deprivation is evaluated. The two most common approaches are selecting either households or individuals as the unit of analysis. The proposed methodology of this article allows selecting either of these two different units. While individual-based measures allow the unmasking of differences in multidimensional deprivation across demographic subpopulation groups, household-based measures conceive households as co-operative units that jointly face the deprivation suffered by the household members.

Considering household as the unit of multidimensional deprivation analysis implies understanding the burden that deprivation places as shared among household members. For example while child mortality refers to a particular episode that is suffered by children, using the household as the unit of deprivation analysis implies that this episode is understood as a phenomenon that not only affects children but also the household as a whole. The living conditions and behaviours of household members contribute to reducing or increasing the frequency of such situations, and the burden of the episode is faced collectively by the household.

An example of a household-based approach to measurement where possible intrahousehold externalities arising from the presence of non-deprived household members is the proposed approach of Basu and Foster (1998) for the case of literacy. In this case, the scholars proposed taking into account in the literacy measurement not only individuals' ability to read and write but also the additional advantage that literate householdsâĂŹ members bring to illiterate members in the household. Extensions have been developed by Subramanian (2004), Subramanian (2008), and Chakravarty and Majumder (2005), and a similar approach but for the case of the unemployment rate has been proposed by Basu and Nolen (2008).

As such, the proposed methodology of this paper enables using either the household or the individual as the unit of analysis. It recognizes that selecting individuals or households have embedded different normative criteria that need to be analysed and defined according to the purposes of each particular application.

Defining needs. Implicit in our approach is that an individual can be regarded as deprived by a particular indicator only if it measures an achievement that can be viewed as something that this individual legitimately needs. Needs differ across dimensions of multidimensional deprivation by population subgroup. For example while adults who do not have work opportunities despite looking for them can be catalogued as employment deprived, children cannot be catalogued as deprived in the absence of employment. Conversely, children under 11 years old who are forced to work would be catalogued as deprived. Children are accountable on other deprivations that are relevant to them, such as access to education services. As such, adults and children have different sets of needs. While adults need access to job opportunities and are considered employment deprived whenever they do not have access to them, children need access to basic school services and are considered educationally deprived if they lack such access.

Needs are, thus, incorporated into my multidimensional deprivation family of indices by excluding from the calculations all dimensions that do not correspond to needs for a particular individual. As such, by setting the applicable population subgroups where relevant to measure presence or absence of deprivation in each well-being indicator, the practitioner formalizes whom is defined as legitimately needing each of these dimensions. These differences in need make visible the normative definition of legitimate and illegitimate differences in achievement levels. Whereas differences in achievement level within the applicable population subgroup of each indicator are set as illegitimate, differences within the non-applicable population subgroup are catalogued as fair and are therefore tackled by the measurement process.

In consequence, setting the applicable populations results in being a key normative decision in our proposed approach. It is suggested that they be made using context-specific norms of what is considered desirable and undesirable in each of the dimensions included within the multidimensional index, or available international indicator definitions.

For example in the education dimension, deprivation indicators may be defined using the ranges of ages suitable to measure enrolment and school lag according to each country. Another example is child labour. The International Labour Organization describes in its regulations the age ranges defined as suitable to measure this kind of deprivation and the activities and time duration that are considered as acceptable for this matter. Still, these definitions are context specific and should be tailored with special care.

Measuring the burden of multidimensional deprivation. The selection of parameters that describe the burden that multidimensional deprivation places over the household also constitutes an important context-specific definition. There is no correct or incorrect selection of parameters. However, each combination produces different household rankings and therefore it can be utilized according to the circumstances.

To select the most appropriate combination of parameters, two different approaches can be employed. First, the combination of parameters can be selected from a normative perspective. While count-based approaches ($\theta = 0$) give either to each dimension (using $\beta = 0$) or to each deprivation ($\beta = 1$) an equal absolute value in the measurement of the burden of multidimensional deprivation, share-based approaches ($\theta = 1$) give an equal absolute value to each household, regardless of their demographic composition and size, and taking into account the possible scale economies that arise at this level. An intermediate normative perspective approach corresponds to a θ parameter in between these two solutions. The value of θ reflects the responsiveness of the burden of deprivation to the scale of need, and values of θ close to zero convey a lower response of the burden of multidimensional deprivation to the size of need. Conversely, values of θ close to one convey a greater response of the burden of deprivation to the size of need.

On the other hand, the second possible course of action corresponds to determining the combination of parameters that enables non-biased societal multidimensional deprivation incidence profiles. This combination of parameters can be obtained, as discussed in Section 4, by simulating a counterfactual scenario of no illegitimate difference in deprivation. The evaluation of the measures in such a scenario enables determination of whether or not a particular combination of β and θ to describe the burden of multidimensional deprivation enables any two households with different sets of needs but no systematic difference in deprivation to be classified as equivalently deprived.

The selection of parametric values of β and θ to describe the burden of multidimensional deprivation under the proposed methodology of this paper is advised in light of robustness checks using different multidimensional deprivation thresholds and specifications.

7 Concluding remarks

This paper proposed a family of multidimensional deprivation indices that takes into account differences in need that demographically heterogeneous units (i.e. either households of different size and composition or individuals of different population subgroups) exhibit. The proposed family of indices is meant to be applicable for the purposes of policy and suitable for contexts where multidimensional deprivation is aimed to be measured through a wide range of indicators that describe differences in needs.

To measure the burden of multidimensional deprivation, we use different approaches to measurement that range from count-based (absolute) to share-based (relative) and intermediate approaches. These different approaches to measurement to sort and identify the multidimensional deprived population produce significantly different multidimensional deprivation profiles. As such, these different approaches were evaluated using the 2013 PHS and counterfactual scenarios of no systematic differences in deprivation across households/individuals but only differences in needs. Measures able to catalogue households/individuals as equivalently deprived in these scenarios are said to be portraying an unbiased multidimensional deprivation profile.

Multidimensional deprivation measures, which do not address differences in needs, as for example the dimensions AF's count-based approach, were found to yield biased multidimensional deprivation incidence profiles. In general, count-based approaches produced larger multidimensional deprivation incidence among households with larger sizes of needs, despite having no systematic difference in deprivation. Share-based approaches, in contrast, produced larger multidimensional deprivation incidence among households with smaller sizes of needs. The degree to which we must account for these differences in need, therefore, stands out as relevant.

To evaluate the robustness of these results, the behaviour of the measures was analysed using different alternative specifications of the index to address differences in needs. Balancing procedures as proposed by Alkire (2015), which imply in each well-being dimension accounting for all population subgroups with one applicable deprivation indicator, were also discussed and evaluated as alternative methodological approaches. The results of this paper proved to be robust under all of these alternative considerations.

In the context of the Paraguayan implemented index example, particular members of our proposed family of measures of the burden of multidimensional deprivation demonstrated the ability to depict as equivalently deprived households with no systematic difference in deprivation but only different sets of needs. They, therefore, confirmed providing an unbiased multidimensional deprivation incidence profile in this specific context.

We also evaluate the proposed family of measures of this paper in terms of its properties. The results of this evaluation demonstrate the proper orientation of the family of measures and the desirable non-sensitivities, so we can conclude our proposed technology is adequate for the purposes of poverty measurement.

However, within our used framework the limitations that a parametric equivalence scale of this type can be recognized. First, considering that the ultimate purpose of a multidimensional measure of deprivation is to capture unfair disadvantage, differences in deprivation due to other fair sources are not accounted. Further research is required to disentangle the effect that other sources of legitimate differences might have over multidimensional deprivation incidence profiles, such as preferences or needs not necessarily based on the still limited number of observable attributes (i.e. household size, composition, or age and gender) that are addressed in this paper. In addition, differences in need are due to be analysed in the context of multidimensional deprivation technologies that take into account the complementarity and substitutability that might arise among dimensions.

Second, as analysed by Pollak and Wales (1979), Fisher (1987), and Blundell and Lewbel (1991), for the one-dimensional equivalence scale case, a household's current demographic composition that leads to differences in need might be driven by previous deprivation status as well. For example a particular household consisting of two adults and five children might be this size not only because both adults have a preference for many children, but also because they did not have access to pregnancy prevention education or could not afford to use some form of birth control. Then, household composition not only reflects needs or preferences, catalogued in this paper as producing fair differences in deprivation among households, but also current household compositions might be a reflection of avoidable and unfair previous states of deprivation. This is a complex issue that is left for further research.

Furthermore, the proposed methodology of this paper measures multidimensional deprivation either at the individual or at the household level without addressing the intrahousehold bargaining power and allocation of resources that might be conducive to ameliorating or intensifying each individual's burden of deprivation. In particular, the one-dimensional welfare comparisons literature has shown that households do not behave as a single unit but rather under collective instances where household's behaviour is the outcome of the joint decisions of its members (Chiappori, 1992; Browning et al., 2013; Mazzocco, 2005). As such, further research is required to provide a multidimensional measurement technology able to account for these intrahousehold differences and their effect on the burden that multidimensional deprivation places on each individual.

Nonetheless, following Elster and Roemer (1991, pp.1), who exhorts any notion of well-being to be based on appropriately operationalized interpersonal comparisons, and also to be adequate for the purposes of distributive justice. The family of measures presented in this paper contributes to the multidimensional deprivation measurement literature by enhancing the comparability across households or individuals that exhibit different needs, as well as being adequate for the purposes of multidimensional deprivation measurement.

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Appendix A. Simulations' pseudo-code

Pseudocode_sim

```
**************************************************************************
* Terminology
**************************************************************************
Dimensions : noseguro saluate noasiste noedu dwelling
Variables
I_*        : Individual deprivation indicators (g_ij)
pr_*       : Applicable population subgroups per indicator (s_ij)
hhid       : Household identification
Scalars
A_*        : Societal observed amount of deprivation
C_*        : Size of the applicable population subgroup
p03        : Household head identificator (one per household)

***************************************************************************
* Counterfactual scenario of no ilegitimate difference in deprivation
***************************************************************************
* The replications of each counterfactual scenario
local repe = 1000
local sim=0
while `sim'<=`repe'-1 {
        * Data at the individual level
        use database, clear

        foreach X in noseguro saluate noasiste noedu {
                * Assigning a random number to individuals within the j-applicable population
                gen random = runiform() if I_`X'!=. & pr_`X'>0 & pr_`X'!=.
                sort random
                gen temp=_n
                * The simulated deprivation distribution
                gen R_`X'=0 if temp<=C_`X' & I_`X'!=. & pr_`X'==1
                replace R_`X'=1 if temp<=A_`X' & I_`X'!=. & pr_`X'==1
                drop temp random
        }
        gen random = runiform() if I_dwelling!=. & pr_dwelling==1 & p03==1
        sort random
        gen temp=_n
        gen temp_dwelling=0 if temp<=C_dwelling & I_dwelling!=. & pr_dwelling==1 & p03==1
        replace temp_dwelling=1 if temp<=A_dwelling & I_dwelling!=. & pr_dwelling==1 & p03==1
        bysort hhid: egen R_dwelling=mean(temp_dwelling)
        drop temp random

        * Calculating measures in the counterfactual scenario
        foreach X in noseguro saluate noasiste noedu dwelling {
                * The beta parameter of deprivation aversion
                foreach b of numlist 0 1 {
                        * Household dimensional deprivation indicator
                        capture drop temp
                        bysort hhid: egen temp=total(R_`X')
                        gen dh_B`b'_j`X'=(temp)^`b'
                        replace dh_B`b'_j`X'=0 if temp<=0
                        * Size of household dimensional needs
                        capture drop temp
                        bysort hhid: egen temp=total(pr_`X')
                        gen nh_B`b'_j`X'=(temp)^`b'
                        replace nh_B`b'_j`X'=0 if temp<=0
                }
        }

        * Data at the household level
        keep if p03==1

        * Size of household multidimensional needs
        egen needs_B1=rowtotal(nh_B1_j*)
        egen needs_B0=rowtotal(nh_B0_j*)

        foreach b of numlist 0 1 {
                * The theta parameter of scale response of deprivation to needs
                foreach Theta of numlist 0(0.01)1 {
                        local t=round(`Theta'*100 ,1)
                        * Household multidimensional deprivation
                        egen m_B`b'T`t'=rowtotal(dh_B`b'_j*)
                        replace m_B`b'T`t'=m_B`b'T`t'/(needs_B`b'^`Theta')
                        replace m_B`b'T`t'=0 if needs_B`b'<=0
                        * Identification of the multidimensionally deprived
                        egen orderB`b'T`t'=rank(m_B`b'T`t'), unique
                        gen p_s40_B`b'T`t'=(orderB`b'T`t'>=3254)
                        * The delta regression coefficient
                        regress p_s40_B`b'T`t' needs_B0
                        sca delta_B`b'T`t'=(_b[needs_B0])
                }
        }
        local sim=`sim'+1
}
```

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Appendix B. Proofs

Here are the proofs of Lemmas 5.1, 5.2, 5.3, 5.4 and the proof of Proposition 5.6.

Proof of Lemma 5.1. The square matrix π is a permutation matrix, since each row and each column has an entry equal to the unity and all the other entries are 0. Interchanging two rows (or two columns) of π and repeating this procedure, we can transform π to the identity matrix $\mathbf{I}_n$ in a finite number $t \geq 0$ of exchanges of rows or columns of π. Thus the determinant of π is equal to $\det(\pi) = (-1)^t$ and π is non-singular. In particular, $\pi^{\mathbf{T}} = \pi^{-1}$, where $\pi^{\mathbf{T}}$ is the transpose matrix of π and thus the inverse π^{-1} of π is permutation matrix as well.

To see this, consider that the 1 entry in the cell $\pi(i, j)$ of π translates to 1 in the cell $\pi^{\mathbf{T}}(j, i)$ of $\pi^{\mathbf{T}}$, since the rows (resp. columns) of the former matrix are the columns (resp. rows) of the latter. Multiplying these matrices, observe that the for the entries of the main diagonal $a_{k,k}$ holds for $k = 1, \ldots, n$ that $a_{k,k} = \sum_{j=1}^{n} \pi_{k,j} \pi^{\mathbf{T}}_{j,k} = 1$ and the other entries are 0. Thus $\mathbf{G}'' = \pi \mathbf{G}'$ permutes the rows of $\mathbf{G}'$. These arguments prove the first and second properties.

The population replication invariance property follows directly by linearity of expectation, since for each positive integer $t > 1$, we add t identical copies of the population and divide their sum by $t \cdot R$. We now prove the "if" direction of the normalization property. Given that $\theta = 1$, by (7) and (10) we have that

$$MD^{\beta,1} = \begin{cases} \dfrac{1}{R} \displaystyle\sum_{k=1}^{R} \dfrac{\sum_{j=1}^{J} d^{\beta}_{kj}}{\left(\sum_{j=1}^{J} n^{\beta}_{kj}\right)} & \text{if } \displaystyle\sum_{j=1}^{J} n^{\beta}_{kj} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since $g_{ij}(s_j) = 1$, for $s_{ij} = 1$ or $g_{ij}(s_j) = 0$, for $s_{ij} = 1$ the result follows. In the opposite direction, using that $0^0 = 1$ and considering (4) and (5) and given that for all household members holds that $g_{ij}(s_j) = 1$ for any i individual in a j dimension with $s_{ij} = 1$, then $\sum_{j=1}^{J} d^{\beta}_{kj} = \sum_{j=1}^{J} n^{\beta}_{kj}$. The case where $g_{ij}(s_j) = 0$ for any i individual in a j dimension with $s_{ij} = 1$ can be handled by a similar argument. The proof of the Lemma is now complete. $\square$

Proof of Lemma 5.2. The multidimensional deprivation focus property is true, since by definition of $MD^{\beta,\theta}$, this measure remains invariant for a change by a constant γ for each person belonging to a non-multidimensionally deprived household. The same reasoning applies as well to the Applicable deprivation focus property. $\square$

Proof of Lemma 5.3. The first two properties are true since adding a positive constant γ to a random variable X and letting μ_X be its average value, we have by the linearity of expectation that

$$\mu_{X+\gamma} = \mu_X + \mu_\gamma = \mu_X + \gamma > \mu_X,$$

given that μ_X is finite. Regarding the dimensional deprivation monotonicity property, assume that $\beta = 1$ and a'_{ij} is obtained from a_{ij} by a dimensional deprivation reduction amongst the multidimensionally deprived, we observe from (7) that $\sum_{j=1}^{J}(d_{kj} + C)^{\beta} \geq \sum_{j=1}^{J} d_{kj}^{\beta}$ if and only if $\beta = 1$, $\theta \in [0,1]$ and where $d' = d_{kj} + C$ and d differ by at most a positive constant C. For if we had $\beta = 0$, then we get zero and the property obviously does not hold. $\square$

Proof of Lemma 5.4. Under the notation in the statements of the above properties, by definition we have

$$MD^{\beta,\theta}(G_\lambda) = \begin{cases} \dfrac{1}{I_\lambda}\displaystyle\sum_{k=1}^{I_\lambda} \dfrac{\sum_{j=1}^{J} d_{kj}^{\beta}}{\left(\sum_{j=1}^{J} n_{kj}^{\beta}\right)^{\theta}} & \text{if } \displaystyle\sum_{j=1}^{J} n_{kj}^{\beta} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

When $\sum_{j=1}^{J} n_{kj}^{\beta} = 0$ both properties can be easily seen to hold, so assume that $\sum_{j=1}^{J} n_{kj}^{\beta} > 0$. We then have

$$\begin{aligned} \sum_{\lambda=1}^{L} \frac{I_\lambda}{I} MD^{\beta,\theta}(G_\lambda) &= \sum_{\lambda=1}^{L} \frac{I_\lambda}{I}\frac{1}{I_\lambda}\sum_{k=1}^{I_\lambda} \frac{\sum_{j=1}^{J} d_{kj}^{\beta}}{\left(\sum_{j=1}^{J} n_{kj}^{\beta}\right)^{\theta}} \\ &= \frac{1}{I}\sum_{\lambda=1}^{L}\left(\sum_{k=1}^{I_\lambda} \frac{\sum_{j=1}^{J} d_{kj}^{\beta}}{\left(\sum_{j=1}^{J} n_{kj}^{\beta}\right)^{\theta}}\right). \end{aligned}$$

For $\beta = \theta = 1$, we have

$$\sum_{\lambda=1}^{L} \frac{I_\lambda}{I} MD^{1,1}(G_\lambda) = \frac{1}{I}\sum_{\lambda=1}^{L}\left(\sum_{k=1}^{I_\lambda} \frac{\sum_{j=1}^{J} d_{kj}}{\sum_{j=1}^{J} n_{kj}}\right)$$

and the above expression for $\sum_{j=1}^{J} d_{kj} = \sum_{j=1}^{J} n_{kj}$ is

$$\frac{1}{I}\sum_{\lambda=1}^{L}\left(\sum_{k=1}^{I_\lambda} 1\right) = 1,$$

which in turn can be easily seen to be equal to $MD^{1,1}(\mathbf{G})$.

For $\beta = \{0,1\}$ and $\theta = 0$ with $\sum_{j=1}^{J} n_{kj} = 0$, considering that $0^0 = 1$,

$$\begin{aligned} \sum_{\lambda=1}^{L} \frac{I_\lambda}{I} MD^{\beta,0}(G_\lambda) &= \frac{1}{I}\sum_{\lambda=1}^{L}\left(\sum_{k=1}^{I_\lambda}\sum_{j=1}^{J} d_{kj}^{\beta}\right) \\ &= MD^{\beta,0}(\mathbf{G}). \end{aligned}$$

The dimensional decomposability property can be proved by a similar argument. □

Proof of Proposition 5.6. For each $\beta \in \{0,1\}$ and $\theta \in [0,1]$, $m^{\beta,\theta}$ is a positive number. The cardinality property now follows if and only if both the numerator and denominator are integer numbers, since a rational number equals to the division of two integers. Regarding the Absolute translation invariance property, the 'only if' part holds, since given that $(m^{\beta,\theta})' = m_h^{\beta,\theta} + \gamma$ for all R households, then by (7) θ must be 0 for the ranking of two households to be constant. On the other direction, if $\theta = 0$, then from (7), we deduce that $m_h^{\beta,0}$ is a linear function. Indeed, for a constant γ, we have that $m_{\gamma\cdot h}^{\beta,0} = \gamma \cdot m_h^{\beta,0}$ and for h, s holds that $m_{h+s}^{\beta,0} = m_h^{\beta,0} + m_s^{\beta,0}$.

The third property follows immediately by considering the linearity of $m_h^{\beta,0}$. Note that $m_h^{\beta,\theta}$ is not linear for $\theta > 0$ and so it holds $m_{\gamma\cdot h}^{\beta,\theta} = m_h^{\beta,\theta}$ for some positive constant γ. □

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