Diewert Discussion of “Aggregate Productivity and the Productivity of the Aggregate: Connecting the Bottom-Up and Top-Down Approaches”

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1. Introduction

- **Balk contrasts two approaches to determining the productivity of an aggregate of production units:**

  (i) the bottom-up approach that takes a weighted average of the micro productivity numbers and

  (ii) the top-down approach that applies the same productivity methodology to the aggregate of the individual production units, generates a formula for aggregate productivity growth and then looks at the resulting formula and we attempt to relate the aggregate result to the micro rates of productivity growth.

- **I will focus on the top-down results because the bottom-up approach has too many degrees of freedom (what type of average should be used, what type of weights should be used, etc.)**

2. The Accounting Framework

- **This section goes over the fundamentals of aggregating over sectors.**
- **Bert shows how intermediate input transactions cancel out and lead to consolidated deliveries and inputs from sectors that are external to the sectors considered within the aggregate that is being considered.**
- **Bert might want to mention Chapters 18 and 19 in the Producer Price Index Manual which shows how the consolidation process works and has a numerical example to illustrate how real value added is constructed over an aggregate of sectors.**

3. Bottom-Up and Top-Down Approaches Connected

- **In this section, the bottom-up and top-down approaches are formally defined.**
- **Basically, the bottom-up approach defines aggregate productivity as a function of the micro productivities and a set of weights.**
- **The top-down approach works with the consolidated data of the set of micro units and defines aggregate productivity using the aggregated data, using the same definitions that were used to define the micro productivity growth rates.**
- **Balk then asks: when are the two approaches equivalent; i.e., can we find a set of weights for the micro productivity aggregation process that will equate the two productivity concepts? A subsidiary question is: if so, are the resulting weights such that they sum to one?**

4. Value Added Based Total Factor Productivity

- **This is a key section and the remainder of the paper more or less depends on the results in this section.**
- **Unfortunately, I could not reproduce the results that were obtained in this section.**
- **My problem is that the author did not provide enough detail for his derivations. He derived his basic result, equation (18), in one page. It took me 5 pages of definitions and computations to derive a counterpart to his equation (18).**
- **I will spend the remainder of my discussion deriving my version of his equation (18).**
- **I may have made a mistake in my computations or I may have misinterpreted his notation so this fact must be kept in mind.**

Setting the Stage: Notation and Definitions

- **We suppose that we have observed the price and quantity data for K production units for two periods (b, the base period and t, the current period).**
- **The vector of net outputs for sector k in time period s is y^{ks} (output components have positive signs, intermediate input components have negative signs) and the corresponding vector of positive prices is p^{ks} for k = 1,...,K and s = b,t.**
- **The vector of positive primary inputs for sector k in time period s is x^{ks} and the corresponding vector of positive primary input prices is w^{ks} for k = 1,...,K and s = b,t.**
- **The *value added* for sector k in period s, V^{ks}, is defined by (1) below and the *primary input cost* for sector k in period s, C^{ks}, is defined by (2) below:**

Basic Definitions

(1) $V^{ks} \equiv p^{ks} \cdot y^{ks}$; $\quad k = 1,\ldots,K;\ s = b,t;$

(2) $C^{ks} \equiv w^{ks} \cdot x^{ks}$; $\quad k = 1,\ldots,K;\ s = b,t.$

- Period s *aggregate value added* V^s and *aggregate primary input cost* C^s for sector k are defined by (3) and (4) below:

(3) $V^s \equiv \Sigma_{k=1}^{K} V^{ks}$; $\quad s = b,t;$

(4) $C^s \equiv \Sigma_{k=1}^{K} C^{ks}$; $\quad s = b,t.$

- We pick our favourite *bilateral index number formula* in order to aggregate output prices in each sector k (of the form $P^k(p^{kb},p^{kt},y^{kb},y^{kt})$) and use it to define the level of net output prices in period t relative to period s as $P^{kt} \equiv P^k(p^{kb},p^{kt},y^{kb},y^{kt})$ for $k = 1,\ldots,K$.

Period t and Period b Sector Definitions (Outputs)

- **We define the *sector k real value added* for period t, Y^{kt}, by deflating the nominal value added for sector k in period t, V^{kt}, by our sectoral price index P^{kt}:**

$$(5)\ Y^{kt} \equiv V^{kt}/P^{kt}\ ; \qquad k = 1,\ldots,K.$$

- **For period b, we set the *real value added of sector k*, Y^{kb}, equal to its nominal value added and set the aggregate price of real value added for sector k in period b, P^{kb}, equal to one; i.e., we have the following definitions:**

$$(6)\ Y^{kb} \equiv V^{kb}\ ; \qquad k = 1,\ldots,K;$$

$$(7)\ P^{kb} \equiv 1\ ; \qquad k = 1,\ldots,K.$$

Period t and Period b Sector Definitions (Inputs)

- **Now repeat the above process for primary inputs. Again, we pick our favourite bilateral index number formula in order to aggregate primary inputs in each sector k (of the form $P^{k*}(w^{kb},w^{kt},x^{kb},x^{kt})$) and use it to define the level of *sector k aggregate input prices in period t* relative to period s as $W^{kt} \equiv P^{k*}(w^{kb},w^{kt},x^{kb},x^{kt})$ for k = 1,…,K. We define the *sector k real primary input* for period t, X^{kt}, by deflating the sector k nominal primary input cost, C^{kt} by our sector k primary input price index W^{kt}:**

$$(8)\quad X^{kt} \equiv C^{kt}/W^{kt}\,; \qquad k = 1,\ldots,K.$$

- **For period b, we set the real primary input of sector k, X^{kb}, equal to its nominal cost of primary inputs and set the aggregate price of real primary input for sector k in period b, W^{kb}, equal to one; i.e., we have the following definitions:**

$$(9)\quad X^{kb} \equiv C^{kb}\,; \qquad k = 1,\ldots,K;$$

$$(10)\quad W^{kb} \equiv 1\,; \qquad k = 1,\ldots,K.$$

Sectoral Value Added, Input and TFP Growth

- **Using the above definitions, we can derive the following expressions for (one plus) *real value added growth* in sector k going from period b to t, Y^{kt}/Y^{kb}, and (one plus) *real primary input growth* in sector k going from period b to t, X^{kt}/X^{kb}:**

(11) $Y^{kt}/Y^{kb} = (1/P^{kt})(V^{kt}/V^{bt})$; $k = 1,\ldots,K$;

(12) $X^{kt}/X^{kb} = (1/W^{kt})(C^{kt}/C^{bt})$; $k = 1,\ldots,K$.

- **Using (11) and (12), (one plus) *Total Factor Productivity (TFP) growth for sector k going from period b to t*, Γ^{kt}, can be defined as (one plus) real value added growth divided by (one plus) real primary input growth:**

(13) $\Gamma^{kt} \equiv [Y^{kt}/Y^{kb}]/[X^{kt}/X^{kb}]$; $k = 1,\ldots,K$.

The Decomposition of Aggregate Value Added Growth

- **We now turn our attention to the decomposition of aggregate nominal value added growth into price and quantity (or volume) components.**
- **We will utilize a general approach recommended by Balk in the present paper:**

"The point of departure of this paper is that aggregate productivity should be interpreted as productivity of the aggregate." Bert M. Balk (2016; 2).

- **We have already defined aggregate value added in periods b and t, V^b and V^t, by equations (3).**
- **We will treat each net output produced by each sector as a contributing net output to aggregate value added for the period under consideration.**
- **Thus the period s price vector is now $[p^{1s},p^{2s},...,p^{Ks}] \equiv p^s$ and the corresponding quantity vector is $[y^{1s},y^{2s},...,y^{Ks}] \equiv y^s$ for s = b,t.**

The Decomposition of Aggregate Value Added Growth (cont)

- **We again pick our favourite bilateral index number formula in order to aggregate the net outputs of the set of sectors that are in scope (this aggregate index has the form $P(p^b,p^t,y^b,y^t)$) and we use it to define the *aggregate level of net output prices in period t* relative to period s as $P^t \equiv P(p^b,p^t,y^b,y^t)$.**
- **We define the *economy wide real value added for period t*, Y^t, by deflating the aggregate nominal value added, V^t by our aggregate price index P^t:**

(14) $Y^t \equiv V^t/P^t = (1/P^t)\, \Sigma_{k=1}^{K} V^{kt} = \Sigma_{k=1}^{K} (P^{kt}/P^t)Y^{kt}$

- **where the last equation in (14) follows from equations (5).**
- **We note that Balk's general approach has been used before to construct national value added price and quantity indexes; e.g., see Chapters 18 and 19 of the IMF, ILO, OECD, UN and World Bank (2004; 464) for a worked example. However, the idea of using an aggregate value added function to aggregate over sectors was noted by Bliss (1975; 146) and many others; see Diewert (1980; 464-465) for additional references to the literature. The paper by Tang and Wang (2004) is also relevant at this point.**

The Decomposition of Aggregate Value Added Growth (cont)

- **For period b, we set *aggregate real value added*, Y^b, equal to the corresponding aggregate nominal value added and set the price of aggregate real value added output in period b, P^b, equal to one; i.e., we have the following definitions:**

(15) $Y^b \equiv V^b = \sum_{k=1}^{K} V^{kb} = \sum_{k=1}^{K} Y^{kb}$;

(16) $P^b \equiv 1$

- **where the last equation in (15) follows using definitions (6).**

- **For future reference, we define *sector k's share of aggregate nominal value added in period b*, s^{kb}, as follows:**

(17) $s^{kb} \equiv V^{kb}/V^b$; $\quad k = 1,\ldots,K$

$= Y^{kb}/Y^b$

- **where the second equation follows from equations (6) and (15).**

Definition of Aggregate Real Value Added Growth

- **We can now calculate (one plus) *the rate of aggregate real value added growth going from period b to t*, Y^t/Y^b, as follows:**

(18) $Y^t/Y^b = [\Sigma_{k=1}^K (P^{kt}/P^t)Y^{kt}]/[\Sigma_{k=1}^K Y^{kb}]$ **using (14) and (15)**

$= [\Sigma_{k=1}^K (P^{kt}/P^t)(Y^{kt}/Y^{kb})Y^{kb}]/[\Sigma_{k=1}^K Y^{kb}]$

$= \Sigma_{k=1}^K s^{kb} (P^{kt}/P^t)(Y^{kt}/Y^{kb})$ **using (17).**

- **Thus in general, the aggregate rate of real value growth, Y^t/Y^b, is not necessarily a weighted average of the sectoral rates of real value added growth, Y^{kt}/Y^{kb}, due to the presence of the output price terms P^{kt}/P^t in equations (18).**
- **Balk's counterpart to our equation (18) is his equation (14)**

(14B) $Y^t/Y^b = \Sigma_{k=1}^K (P^{kt}/P^t)(Y^{kt}/Y^{kb})$

- **which is different from our equation (18).**

Definition of Aggregate Primary Input Growth

- **The above algebra can be repeated for the cost side. We will treat each primary input used by each sector as a contributing input to aggregate real input for the period under consideration. Thus the period s input price vector is now $[w^{1s},w^{2s},\ldots,w^{Ks}] \equiv w^s$ and the corresponding input quantity vector is $[x^{1s},x^{2s},\ldots,x^{Ks}] \equiv x^s$ for s = b,t. We pick our favourite bilateral index number formula in order to aggregate all primary inputs used by the K sectors (of the form $P^*(w^b,w^t,x^b,x^t)$) and use it to define the *aggregate level of primary input prices in period t* relative to period s as $W^t \equiv P^*(w^b,w^t,x^b,x^t)$. We define the *economy wide real input for period t*, X^t, by deflating the aggregate nominal input cost, C^t by our aggregate input price index W^t:**

(19) $$X^t \equiv C^t/W^t = (1/W^t)\,\Sigma_{k=1}^{K} C^{kt} = \Sigma_{k=1}^{K} (W^{kt}/W^t)X^{kt}$$

- **where the last equation in (19) follows from equations (8).**

Definition of Aggregate Primary Input Growth (cont)

- **For period b, we set *aggregate real input*, X^b, equal to the corresponding aggregate nominal input cost and set the price of aggregate real primary input in period b, W^b, equal to one; i.e., we have the following definitions:**

(20) $X^b \equiv C^b = \Sigma_{k=1}^{K} C^{kb} = \Sigma_{k=1}^{K} X^{kb}$;

(21) $P^b \equiv 1$

- **where the second equation in (20) follows using definitions (9).**

- **For future reference, we define *sector k's share of aggregate nominal input cost in period s*, σ^{ks}, as follows:**

(22) $\sigma^{ks} \equiv C^{ks}/C^s$; $\quad$ **k = 1,...,K; s = b,t.**

Definition of Aggregate Primary Input Growth (cont)

- **When s = b, σ^{kb} can be written as a *sectoral real input share* as well as a sectoral nominal input cost share; i.e., using equations (9), (20) and (22), we have:**

(23) $\sigma^{kb} \equiv X^{kb}/X^{b}$; $\qquad$ **k = 1,…,K.**

- **We can now calculate (one plus) *the rate of aggregate real primary input growth going from period b to t*, X^{t}/X^{b}, as follows:**

(24) $X^{t}/X^{b} = [\Sigma_{k=1}^{K} (W^{kt}/W^{t})X^{kt}]/[\Sigma_{k=1}^{K} X^{kb}]$ $\qquad$ **using (19) and (20)**

$= [\Sigma_{k=1}^{K} (W^{kt}/W^{t})(X^{kt}/X^{kb})X^{kb}]/[\Sigma_{k=1}^{K} X^{kb}]$

$= \Sigma_{k=1}^{K} \sigma^{kb} (W^{kt}/W^{t})(X^{kt}/X^{kb})$ $\qquad$ **using (23).**

- **Thus in general, the aggregate rate of real input growth, X^{t}/X^{b}, is not necessarily a weighted average of the sectoral rates of real input growth, X^{kt}/X^{kb}, due to the presence of the input price terms W^{kt}/W^{t} in the last line of (24).**

Defining the Period t and b Productivity Levels

- **We define (one plus) *aggregate productivity growth* going from period b to t, Γ^t, as (one plus) aggregate output growth, Y^t/Y^b, divided by (one plus) aggregate input growth, X^t/X^b:**

$$(25)\ \Gamma^t \equiv [Y^t/Y^b]/[X^t/X^b] = [Y^t/X^t]/[Y^b/X^b].$$

- **Using (14) and (19), we can obtain the following expression for Y^t/X^t, the period t productivity level:**

$$\begin{aligned}(26)\ Y^t/X^t &= [\Sigma_{k=1}^K (P^{kt}/P^t)Y^{kt}]/[\Sigma_{k=1}^K (W^{kt}/W^t)X^{kt}] \\ &= [\Sigma_{k=1}^K (P^{kt}/P^t)(W^t/W^{kt})(Y^{kt}/X^{kt})(W^{kt}/W^t)X^{kt}]/[\Sigma_{k=1}^K (W^{kt}/W^t)X^{kt}] \\ &= [\Sigma_{k=1}^K (P^{kt}/P^t)(W^t/W^{kt})(Y^{kt}/X^{kt})(C^{kt}/W^t)]/[\Sigma_{k=1}^K (C^{kt}/W^t)] \quad \text{using equations (8)} \\ &= \Sigma_{k=1}^K (P^{kt}/P^t)(W^t/W^{kt})(Y^{kt}/X^{kt})\sigma^{kt} \quad \text{using definitions (4) and (22).}\end{aligned}$$

Defining the Period t and b Productivity Levels (cont)

- **Similarly, using equations (15) and (20), we can obtain the following expression for Y^b/X^b:**

(27) $Y^b/X^b = [\Sigma_{k=1}^K Y^{kb}]/[\Sigma_{k=1}^K X^{kb}]$

$= [\Sigma_{k=1}^K (Y^{kb}/X^{kb})X^{kb}]/[\Sigma_{k=1}^K X^{kb}]$

$= \Sigma_{k=1}^K (Y^{kb}/X^{kb})\sigma^{kb}$ **using definitions (4), (20) and (23).**

- **Thus the base period aggregate productivity level, Y^b/X^b, is a share weighted average of the sectoral base period productivity levels, Y^{kb}/X^{kb}, where the sector k weight is σ^{kb}, the sector k share of total primary input cost in period t.**

A Preliminary Result

- **Before we take the ratio of (26) to (27), we need a preliminary result. We need to calculate the following ratio for k = 1,...,K:**

(28) $\sigma^{kb}(Y^{kb}/X^{kb})/\Sigma_{j=1}^{K}\ \sigma^{jb}(Y^{jb}/X^{jb})$

$= (X^{kb}/X^{b})(Y^{kb}/X^{kb})/[\Sigma_{j=1}^{K}\ (X^{jb}/X^{b})(Y^{jb}/X^{jb})]$

using definitions (23)

$= (Y^{kb}/X^{b})/[\Sigma_{j=1}^{K}\ (Y^{jb}/X^{b})$

$= Y^{kb}/\Sigma_{j=1}^{K}\ Y^{jb}$

$= s^{kb}$ **using definitions (17).**

- **Recall that s^{kb} is the sector k's share of base period aggregate value added.**

Our Final Formula for Aggregate TFP Growth

- **Now take the ratio of (26) to (27) and we obtain the following expression for aggregate value added productivity growth:**

(29) $\Gamma^t = [Y^t/X^t]/[Y^b/X^b]$

$= [\Sigma_{k=1}^K (P^{kt}/P^t)(W^t/W^{kt})(Y^{kt}/X^{kt})\sigma^{kt}]/[\Sigma_{k=1}^K (Y^{kb}/X^{kb})\sigma^{kb}]$

$= \Sigma_{k=1}^K (P^{kt}/P^t)(W^t/W^{kt})[(Y^{kt}/X^{kt})/(Y^{kb}/X^{kb})](\sigma^{kt}/\sigma^{kb})(Y^{kb}/X^{kb})\sigma^{kb} / \Sigma_{j=1}^K (Y^{jb}/X^{jb})\sigma^{jb}]$

$= \Sigma_{k=1}^K (P^{kt}/P^t)(W^t/W^{kt})[(Y^{kt}/X^{kt})/(Y^{kb}/X^{kb})](\sigma^{kt}/\sigma^{kb})s^{kb}$ **using (28)**

$= \Sigma_{k=1}^K s^{kb}(P^{kt}/P^t)(W^t/W^{kt})(\sigma^{kt}/\sigma^{kb})\Gamma^{kt}$ **using (13).**

- **Thus (one plus) the aggregate TFP growth rate going from period b to t, Γ^t, is equal to a share weighted average of (one plus) the sectoral TFP growth rates, Γ^{kt}, times growth rate factors reflecting output price change (the P^{kt}/P^t), primary input reciprocal price change (the W^t/W^{kt}) and input cost share change (the σ^{kt}/σ^{kb}).**

Our Final Formula for Aggregate TFP Growth (cont)

- **Note that our formula (29) seems to be considerably different from the counterpart Balk result, which is:**
- **Balk's counterpart to our equation (29) is his equation (18B)**

(18B) $\Gamma^t = \Sigma_{k=1}^{K} (P^{kt}/P^t)(X^{kt}/X^t)\Gamma^{kt}$

- **which is quite different from our equation (29).**
- **In fact, (29) is precisely the expression that Diewert (2015) obtained for aggregate TFP growth as functions of sectoral TFP growth rates.**
- **Thus using Balk's basic framework, we seem to obtain the formula derived by Diewert.**
- **Diewert (2016) used the same framework and obtained some interesting approximations to the exact formula (29).**
- **It would be appropriate to acknowledge the contributions of Tang and Wang (2004) who started this line of inquiry.**

Some References

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